

Testing Continuous Time Models of Interest Rates

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Preliminary Reading

- Yacine Ait-Sahalia (1996), "Testing Continuous-Time Models of the Spot Interest Rate", *The Review of Financial Studies*, 9, 385-426.
- Stachurski, J. and Martin, V.L. (2008), "Computing the Distributions of Economic Models via Simulation", *Econometrica*, 76, 443-450.

Introduction

- Often interested in short rates (pricing bonds and derivatives, monetary policy).
- In modelling interest rates a common starting point is the SDE

$$dr_t = \mu(r_t; \alpha) dt + \sigma(r_t; \beta) dB_t$$

where μ is drift, σ volatility, α and β are parameters, dB is Brownian.

- Interested in the
 - Transitional distribution: $p = p(r | r_t, t)$
 - Marginal distribution: $f_t = f(r_t)$
 - Stationary distribution: $f_\infty = \lim_{t \rightarrow \infty} f_t$
- In many cases closed form solutions do not exist.
- Need to resort to numerical methods (nonparametrics, simulation, approximations)

- 1 SMLE (Hurn, Lindsay and Martin (2003))
The SMLE did not work well as a result of leakage at zero. Clements, Hurn and Lindsay (2003) proposed one solution. Also see Pagan and Martin (1999) (EMM estimator).
- 2 SMM (Conley et al (1997)) proposed an omnibus nonlinear model

$$\text{Drift: } \mu = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^{-1}$$

$$\text{Diffusion: } \sigma^2 = \beta_0 + \beta_1 r + \beta_2 r^{\beta_3}$$

- 3 For a recent review of SDE estimation methods, see Hurn, Jeisman and Lindsay (2007).
- 4 Nonparametric testing (Ait Sahalia (1996)).
Pritsker (1998) found the test has a size $> 50\%$ due to persistence in the data. All models rejected except omnibus model.
- 5 Review of interest rate models given in Pagan, Hall and Martin (1996).

① Levels effect ($\gamma \neq 0$)

The levels effect model of interest rates is

$$dr_t = \kappa (\theta - r_t) dt + \sigma r_t^\gamma dB_t$$

where κ is the adjustment parameter, θ is the long-run level of interest rates, σ controls the stochastics, γ is the levels effect parameter and $dB_t \sim N(0, dt)$. Strong empirical evidence that $\gamma > 1$.

② Cox, Ingersoll, Ross (CIR) ($\gamma = 0.5$).

③ Vasicek ($\gamma = 0$)

An Ornstein-Uhlenbeck process, with conditional moments

$$\text{Mean:} \quad E[r | r_0] = r_0 e^{-\kappa t} + \theta (1 - e^{-\kappa t})$$

$$\text{Covariance:} \quad \text{Cov}(r_s, r_t) = \frac{\sigma^2}{2\kappa} e^{-\kappa(s+t)} \left(e^{-2\kappa \min(s,t)} - 1 \right)$$

Interested in doing nonparametrics, but need to think about

- 1 Issue of leakage at zero
 - Hurn, Lindsay and Martin (2003) identified this problem.
 - Choice of the kernel (circumvent zero leakage).
- 2 Imprecise estimates in the tails of the distribution.
 - Important in VaR
 - Chapman and Pearson (2000) found the Conley et al (1997) results to be spurious due to imprecise estimates in the tails of the distribution.
- 3 Distribution theory: Nonparametrics convergence of a lower order than \sqrt{T} .
- 4 Nonparametrics has slower convergence as the dimension increases.
- 5 Choice of bandwidth is fundamental.
- 6 Persistence in the data invalidate Ait Sahalia asy. distribution (Pristsker (1998)).

An alternative method is adopted to circumvent these problems.

1. Estimating the marginal distribution

Let $r_0 = \{r_{0,1}, r_{0,2}, \dots, r_{0,N}\}$ be N draws from the initial distribution $f_0 = f(r_0)$. The estimated marginal distribution at $t = 1$ is

$$\hat{f}(r_1) = \frac{1}{N} \sum_{i=1}^N p(r_1 | r_{0,i})$$

which is an average over the transition probs. If most values of r_0 are “near” a particular value of r_1 , it will receive a high probability.

The estimated marginal distribution at $t = 2$ is

$$\hat{f}(r_2) = \frac{1}{N} \sum_{i=1}^N p(r_2 | r_{1,i})$$

where $\{r_{1,1}, r_{1,2}, \dots, r_{1,N}\}$ are N observations drawn from

$$dr_{1,i} = \mu(r_{0,i}) dt + \sigma(r_{0,i}) dB_i.$$

The idea is that the model provides the transition probs, hence the distribution the next period.

2. Estimating the stationary distribution

$$\hat{f}(r) = \frac{1}{T} \sum_{t=1}^T p(r|r_t)$$

given a time series on interest rates

$$\{r_1, r_2, \dots, r_T\}.$$

These methods are known as “Look Ahead Estimators” (Henderson and Glynn (2001))

Simple Examples

Discrete AR(1) Model of Interest Rates

- Consider the model

$$r_t = \alpha_0 + \alpha_1 r_{t-1} + \sigma u_t$$

where $\alpha_0 = 0.3$, $\alpha_1 = 0.5$, $\sigma = 0.1$, and $u_t \sim N(0, 1)$.

- The initial distribution ($t = 0$) is a mixture of normals

$$f_0 = \frac{1}{3} (g_1 + g_2 + g_3)$$

where g_i are normal with respective means $\{5, 10, 20\}$ and variances $\{1, 1, 1\}$.

- The transitional distribution at t is normal

$$p(r | r_{t-1}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{1}{2\sigma^2} (r - \alpha_0 - \alpha_1 r_{t-1})^2 \right]$$

To compute the simulation marginal density estimator for $t = 2$ the steps are

- 1 Simulate $N = 100$ observations from f_0 , define as

$$r_0 = \{r_{0,1}, r_{0,2}, \dots, r_{0,N}\}$$

- 2 Simulate $N = 100$ observations from p_1 using

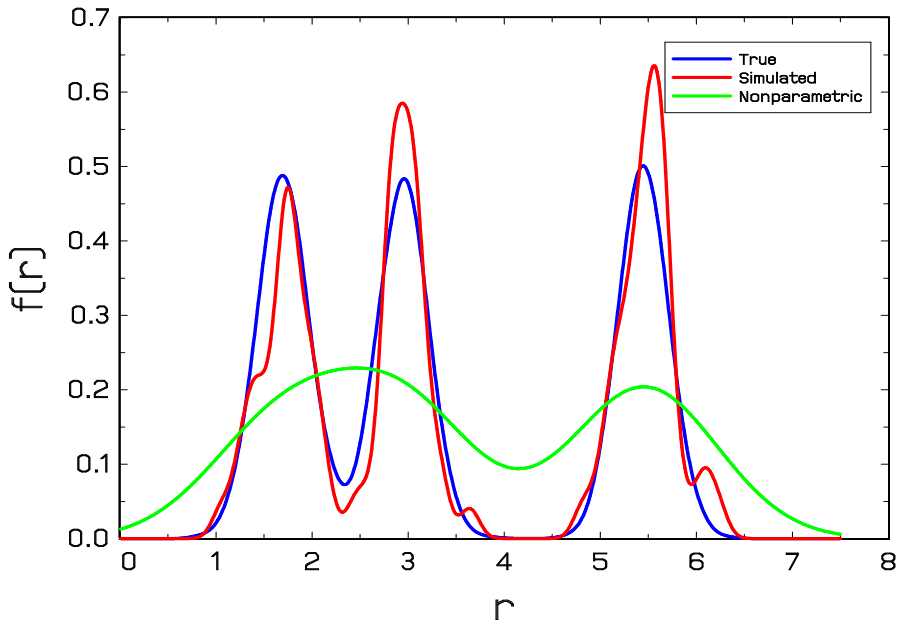
$$r_{1,i} = \alpha_0 + \alpha_1 r_{0,i} + \sigma u_i$$

- 3 For $t = 2$, compute

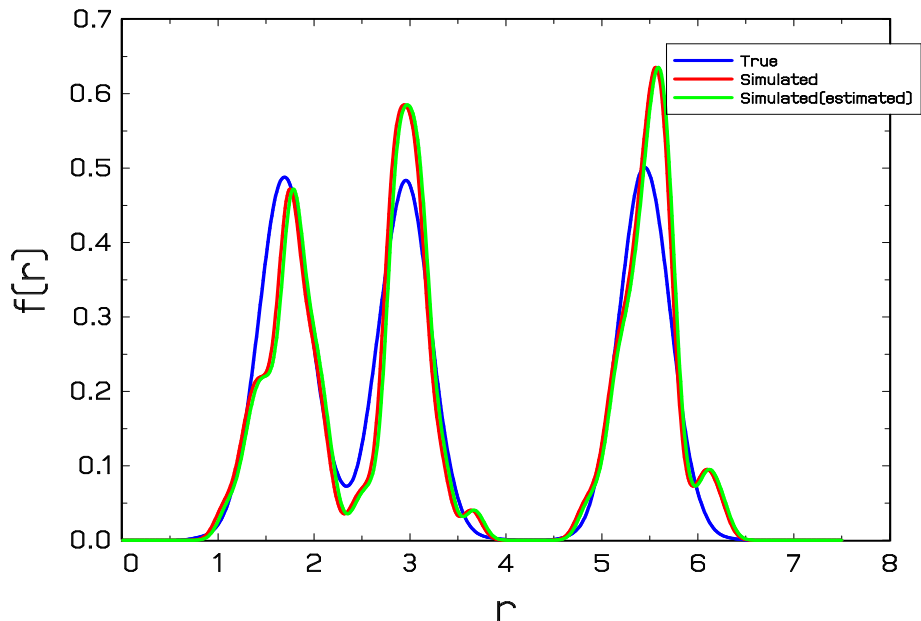
$$\hat{f}(r) = \frac{1}{N} \sum_{i=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{1}{2\sigma^2} (r - \alpha_0 - \alpha_1 r_{1,i})^2 \right]$$

using a grid of values of r .

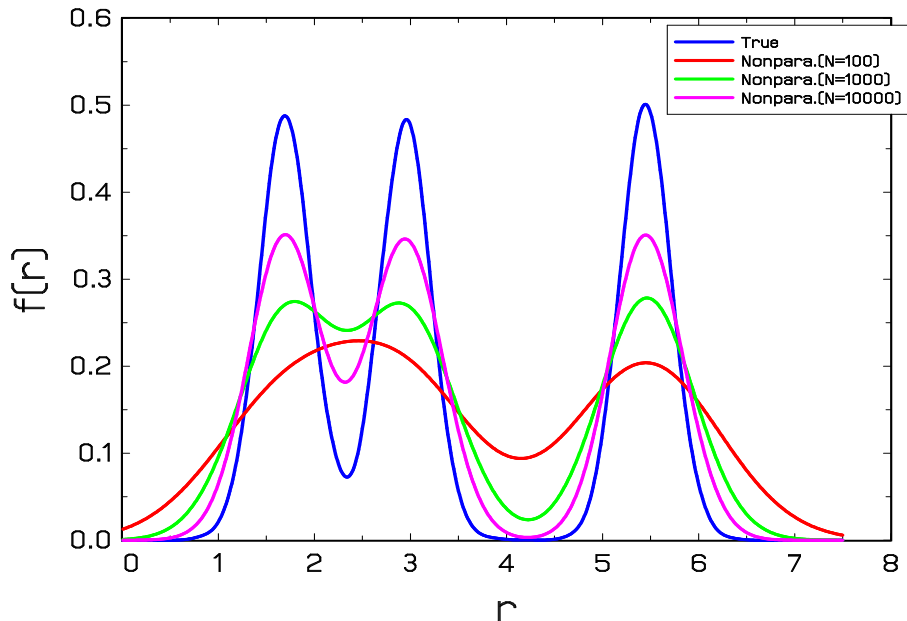
Simulation estimator based on population parameter values.



Simulation estimator based on estimated parameter values (OLS).



Asymptotics slow for the nonparametric method (based on ROT)



Simple Examples

Discrete Logarithmic AR(1) Model of Interest Rates

- This example highlights the problem of leakage with the nonparametric estimator.
- Consider the model

$$\ln r_t = \alpha_0 + \alpha_1 \ln r_{t-1} + \sigma u_t$$

with $\alpha_0 = -0.916$, $\alpha_1 = 0.3$, $\sigma = 0.11$, $u_t \sim N(0, 1)$. $N = 100$ draws.

- The initial distribution ($t = 0$) is a mixture of log-normal distributions

$$f_0 = p_0 = \frac{1}{3} (g_1 + g_2 + g_3)$$

where g_i are lognormal with respective parameters $\mu = \{-7, 3, 7\}$ and $\sigma = \{2, 1, 0.5\}$.

- The transitional distribution of r_t given r_{t-1} is lognormal

$$p(r) = \frac{1}{r\sqrt{2\pi\sigma^2}} \exp \left[-\frac{1}{2\sigma^2} (\ln r - \alpha_0 - \alpha_1 \ln r_{t-1})^2 \right]$$

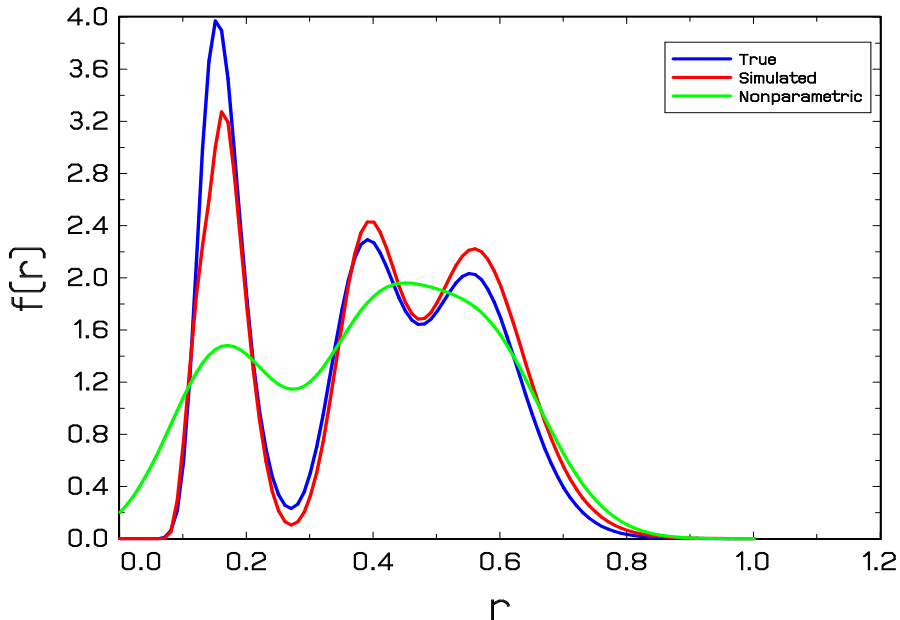
- The simulation density estimator at $t = 2$ is

$$\hat{f}(r) = \frac{1}{N} \sum_{i=1}^N \frac{1}{r\sqrt{2\pi\sigma^2}} \exp \left[-\frac{1}{2\sigma^2} (\ln r - \alpha_0 - \alpha_1 \ln r_{1,i})^2 \right]$$

which is computed for a grid of values of r .

- Two important advantages of the simulation estimator are
 - Automatically leads to an “appropriate” choice of the density.
 - Avoids leakage problems ie positive probability of negative interest rates, as the following diagram demonstrates.

Nonparametric shows evidence of leakage at $r = 0$.



Simple Examples

Comparison with Nonparametrics

- Consider the stationary density of the first interest rate example

$$\hat{f}(r) = \frac{1}{T} \sum_{t=1}^T \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{1}{2\sigma^2} (r - \alpha_0 - \alpha_1 r_t)^2 \right]$$

- The nonparametric density based on a normal kernel and bandwidth h , is

$$\hat{g}(r) = \frac{1}{T} \sum_{t=1}^T \frac{1}{\sqrt{2\pi h^2}} \exp \left[-\frac{1}{2h^2} (r - r_t)^2 \right]$$

- The two are equivalent if
 - r_t is a random walk ($\alpha_1 = 1$) without drift ($\alpha_0 = 0$)
 - the bandwidth is chosen as $h = \sigma$.
- The stationary density form is \sqrt{T} consistent which is not affected by the dimension of the problem. The nonparametric density estimator has slower convergence with the rate of convergence falling as the dimension of the problem increases

Testing Continuous Time Models

- Consider the model

$$dr_t = \kappa (\theta - r_t) dt + \sigma r_t^\gamma dB_t$$

The hypotheses are

$$H_0 : \gamma = 0 \quad (\text{Vasicek})$$

$$H_1 : \gamma \neq 0 \quad (\text{Level})$$

Under the null there is no levels effect in interest rates (Vasicek model).

- Want to construct a statistic to test between these hypotheses using the simulation estimator as have closed form solutions for
 - Transitional density p
 - Stationary density f
- The aim is to generate better power than the nonparametric approach.

- The transitional density under the null is normal

$$p(r|r_t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp \left[-\frac{1}{2\sigma_t^2} (r - \mu_t)^2 \right]$$

with resp. mean and variance

$$\mu_t = \theta + (r_t - \theta) e^{-\kappa\Delta}, \quad \sigma_t^2 = \frac{\sigma^2 (1 - e^{-2\kappa\Delta})}{2\kappa}$$

- The stationary density under the null is normal

$$f_0(r) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp \left[-\frac{1}{2\sigma_0^2} (r - \mu_0)^2 \right]$$

with resp. mean $\mu_0 = \theta$, and variance $\sigma_0^2 = \sigma^2 / 2\kappa$.

- Under the alternative hypothesis for $\gamma = 0.5$, the stationary density is gamma, with $\alpha = 2\kappa\theta / \sigma^2$, $\beta = \sigma^2 / 2\kappa$

$$f_1(r) = \frac{1}{\Gamma(\alpha) \beta^\alpha} r^{\alpha-1} \exp \left[-\frac{r}{\beta} \right]$$

Testing Continuous Time Models

Test Statistic

- The aim is to construct a test that measures the deviation between $f_0(r)$ and the stationary simulation estimator based on the Vasicek model

$$\hat{f} = \frac{1}{T} \sum_{i=1}^T p(r | r_t)$$

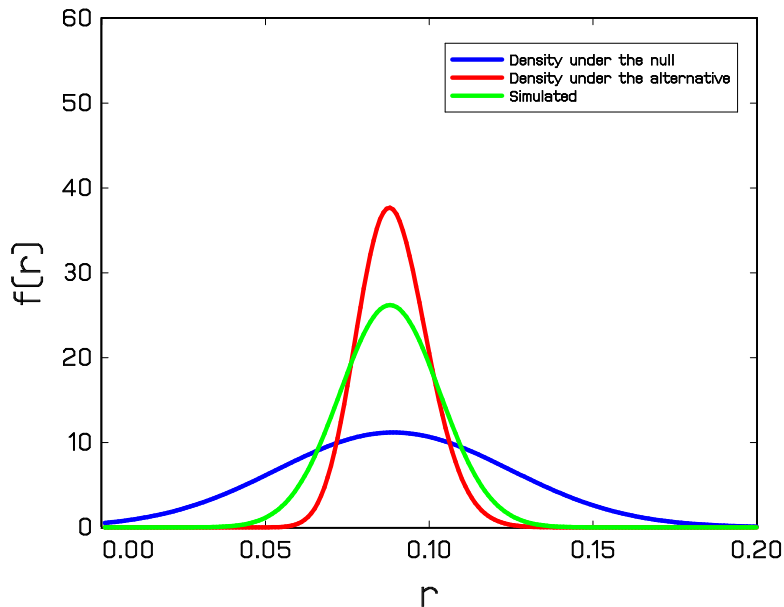
- The proposed statistic is

$$S = T \int \left(\hat{f}(r) - f_0(r) \right)^2 f_0(r) dr$$

which measures the distance between the $\hat{f}(r)$ and $f_0(r)$ weighted by the Vasicek stationary density.

- For the test to have power, \hat{f} should deviate from the Vasicek stationary distribution $f_0(r)$, ie \hat{f} needs to deviate from a false null even though it is evaluated under the null.

Stationary density under the null f_0 ($\gamma = 0$) and alternative f_1 ($\gamma = 0.5$).



Testing Continuous Time Models

Parameter estimation

- The Vasicek model has the exact discretisation

$$r_t = \alpha + \beta r_{t-1} + \nu z_t$$

where

$$\alpha = \theta \left(1 - e^{-\kappa dt}\right)$$

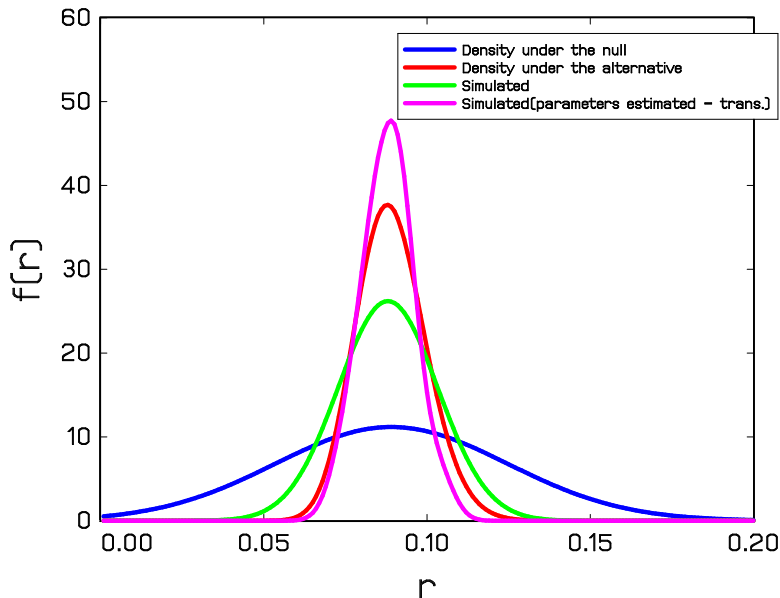
$$\beta = e^{-\kappa dt}$$

$$\nu^2 = \frac{\sigma^2 (1 - e^{-2\kappa dt})}{2\kappa}$$

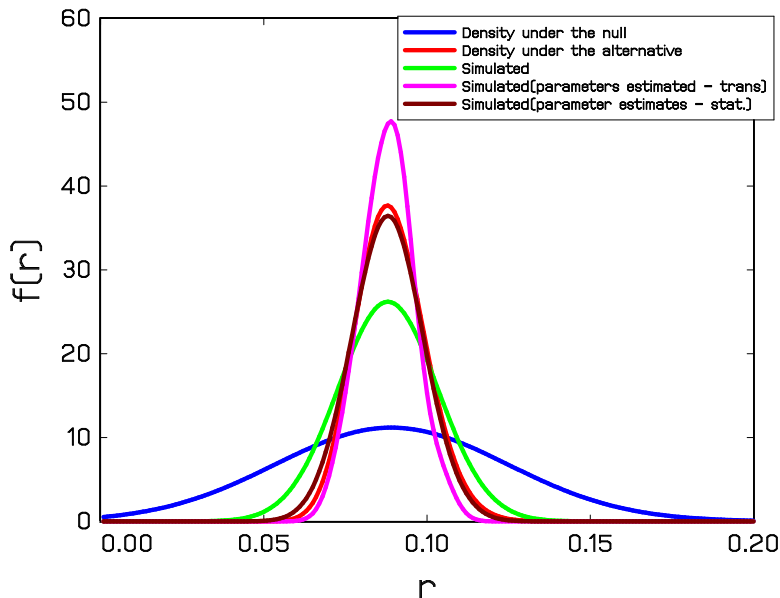
$$z_t \sim N(0, 1)$$

- Can be used to estimate θ, κ, σ^2 from time series data on r_t .
- Two methods used to estimate σ^2 :
 - Transitional form
 - Stationary form

Simulated estimator based on estimating an AR(1) model by OLS (transitional density form)



Simulated estimator based on estimating an AR(1) model by OLS except $\hat{\theta} = \bar{r}$, $\hat{\sigma}^2 = \text{var}(r_t - r_{t-1}) / dt$ (stationary density form).



Testing Continuous Time Models

Asymptotic Distribution

Theorem

$\sqrt{T} (\hat{f} - f_0)$ converges in distribution to a centered Gaussian random variable with covariance function

$$\begin{aligned}\Gamma(r, r') &= \int p(r | r_0, t = 1) p(r' | r_0, t = 1) f(r_0) dr_0 - f(r) f(r') \\ &+ \sum_{t \geq 1} \int p(r | r_0, t = 1) p(r' | r_0, t) f(r_0) dr_0 - f(r) f(r') \\ &+ \sum_{t \geq 1} \int p(r' | r_0, t = 1) p(r | r_0, t) f(r_0) dr_0 - f(r) f(r')\end{aligned}$$

Note that the asy. distribution takes into account persistence.

Corollary

If λ_l are the eigenvalues of Γ and Z_l are independent $N(0, 1)$, then

$$S \xrightarrow{d} \sum_{l=1}^{\infty} \lambda_l Z_l^2$$

Comments.

Note that the variance-covariance estimator takes into account the correlation in the “density”: This is the density analogue of the Newey-West estimator.

This is a more general estimator of autocorrelation as it looks at the correlations in all moments of the distribution and not just second moments as in Newey-West. □

Testing Continuous Time Models

Size of the Test

- The size of the test is computed using

$$dr_t = \kappa (\theta - r_t) dt + \sigma dB_t$$

with $\kappa = 0.858$, $\theta = 0.089$, $\sigma^2 = 0.002$ (based on Ait Sahalia (1996)).

- Finite sample distribution computed using Monte Carlo methods based on 5,000 draws.
- The data are treated as monthly in which case $dt = 1/12$, and the base sample size is $T = 22/dt = 264$.

Sample	True	Estimated params (transitional form)	Estimated params (stationary form)
$T = 264$	4.120%	8.060%	7.120%
$T = 264 \times 2$	4.660%	7.800%	6.860%
$T = 264 \times 3$	4.800%	7.680%	6.620%
Asymptotic	5.000%	5.000%	5.000%

Testing Continuous Time Models

Power of the Test

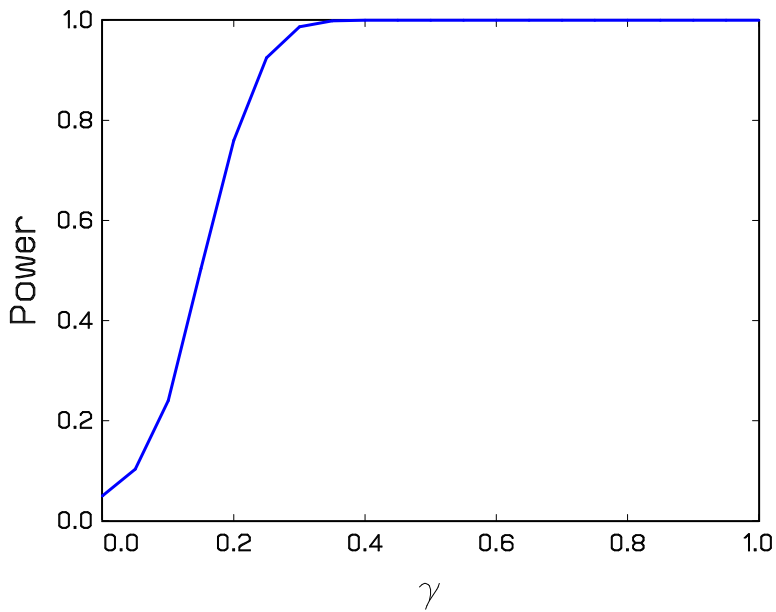
- The power of the test is computed for deviations from $\gamma = 0$, using the levels effect model

$$dr_t = \kappa (\theta - r_t) dt + \sigma r_t^\gamma dB_t$$

- The parameter values and sample size are as above.
- The levels effect parameter has values

$$\gamma = \{0.00, 0.05, 0.10, \dots, 1.00\}$$

- Power functions are adjusted for size of 5%.
- The results show monotonic power functions which achieve power of 1.0 for $\gamma \geq 0.4$



- A new method for testing the properties of interest rates was proposed.
- An important feature of the approach is the incorporation of “parametric” information when constructing the test statistic.
- The new approach circumvents many of the problems from using nonparametric estimation methods.
- Whilst the work is at an early stage, the results look very promising!