

# Treasure game\*

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## Abstract

We study a R&D race between  $n \geq 1$  players where the prize value is known, but the search costs are unknown ex ante. The race is modeled as a multistage game with observed previous actions. A player's probability of success in the current period depends only on his own search in this period.

We provide a complete characterization of the efficient *symmetric Markov perfect equilibrium* (SMPE) for any  $n \geq 1$ . There are two types of inefficiency in search for  $n \geq 2$  players in comparison with the monopolist,  $n = 1$ : a tragedy of commons (for small races) and a free riding (for big races). We also identify several puzzles of search behavior. All puzzles demonstrate that there is no monotonicity if  $n \geq 2$ : players can be better off if the race is longer even though such a race is more costly.

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## 1 Introduction

The R&D literature has grown up substantially in the recent years. It has three main directions.

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(i) The classical papers, Loury (1979), Dasgupta and Stiglitz (1980a, b), Lee and Wilde (1980), assume that each firm in R&D competition makes once-and-for-all expenditure which determines the winner.

(ii) Reinganum (1981, 1982) considers a dynamic R&D race where each firm chooses a time path of expenditures. However, since the author uses the exponential success function the knowledge acquired in the past does not change the probability of the current success in the race. As a result, the equilibrium strategies are independent of time.<sup>1</sup>

(iii) Harris and Vickers (1985) analyze a race model where the winner is the first player to reach the finish line. Fershtman and Rubinstein (1997) consider an interactive model in which two players search for a single hidden treasure in one of a given set of labeled boxes. In both models the players know the upper bound of the costs: the distance in the first model and the number of boxes in the second model.

In this paper, we extend directions (ii) and (iii). In particular, we analyze a dynamic model where  $n$  players search for a treasure hidden somewhere on an island. The value of the treasure is a common knowledge, but search costs are unknown. Each player can observe what the other players have searched in all previous periods. As a result, each period, the size of the island shrinks to the previous-period size minus the part that has been searched by the all players in the previous period.

The game we consider has Schelling's<sup>2</sup> "conflict of partnership and competition" property: players are naturally competing against each other each period, but each player benefits from the other players' previous periods unsuccessful search, because it increases his chance to find the treasure in the current period.

We assume that players are searching different parts of the island and only one player can obtain the treasure. If several players find the treasure simultaneously (search the same part of the island), each of them incurs costs but the treasure will be destroyed (players do not get any treasure). This assumption is standard in the R&D literature (see Chatterjee and Evans, 2004). It can be justified on the ground

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<sup>1</sup>See Reinganum (1989) for more detail discussion about (i) and (ii).

<sup>2</sup>See Schelling (1971).

that if several players discover the treasure simultaneously, they will be involved in the fierce competition afterwards and run out of any surplus. A good example of this situation for just two players is 1960s Lockheed and Douglas jet development competition.<sup>3</sup> Many examples of simultaneous discoveries in science can be found in Merton (1973).

The game we analyze is stochastic where each state is described by the current island size. We restrict our attention to individual Markov strategies: search decision depends only on the current island size. First, we describe a procedure how to construct a *symmetric Markov perfect equilibrium* (SMPE) for any  $n \geq 1$ . We construct the Bellman equation for the problem and use the value iteration method to solve it.

It turns out that there are three types of SMPE: for  $n = 1$ ,  $n = 2$ , and  $n \geq 3$ . We found that each SMPE (for a fixed discount factor,  $\delta$ , and a fixed number of players,  $n$ ) is a *spline* of degree one.<sup>4</sup> It means that our approach also describes the maximal number of search periods which is required to find the treasure. This number is equal to the number of pieces in the spline. We characterize all *knots* of the spline.<sup>5</sup>

Since we are able to describe the optimal searching strategy for a monopolist,  $n = 1$ , we can compare this strategy with the SMPE strategy for  $n \geq 2$ . It turns out that if the initial island size is “very small” (the monopolist will search this island in just one period), players behave as a cartel and search the island in one period. This is the efficient search. It is intuitively clear that if the island size is “too big,” players will not search the island at all. There is an inefficiency here: there are some island sizes such that the monopolist will search them, but  $n \geq 2$  players will not.

If the initial island size is between “very small” and “too big,” then we find that two effects take place. First, if the island size is “slightly bigger” than “very small,” players search the island too fast (in comparison with the efficient search when  $n = 1$ ) in just one period. This effect is the *tragedy of the commons*. Second, if the island

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<sup>3</sup>For more detail see The Economist, 1985; and Chatterjee and Evans, 2004.

<sup>4</sup>A spline is a special function defined piecewise by polynomials. See Ahlberg, Nielson, and Walsh (1967).

<sup>5</sup>A knot between piece  $k$  and  $k + 1$  is the initial island size such that players are indifferent between searching the island in  $k$  or  $k + 1$  periods.

size is “slightly smaller” than “too big,” players search the island too slow: they hope that nobody finds the treasure right now and it will be less costly to search in the next period. This is a *free riding* effect. It is interesting to note that for small discount factors there exists a unique island size where the two effects cancel each other out and search with  $n \geq 2$  players is *always* efficient: players search the island in at most two periods. The cost of search of this island is equal to exactly the treasure value. At this island, all players are getting zero expected payoff in the first search period. Therefore, they leave the efficient island size for the second search period. This observation holds for all  $n \geq 1$ .

Since in the SMPE all players make the same decisions simultaneously, all players have equal probabilities to find the treasure at any period. Therefore, it looks natural to conjecture that a smaller island (smaller costs) is better than a bigger island for all players (players have higher values). It turns out that this conjecture does not hold. As we illustrate in the Example, players might be better off if the island size is bigger. This surprising observation (we call it *Puzzle 1*) means that potentially higher costs might be better off for all players. The puzzle has a similar intuitive explanation as the *tragedy of the commons*: if the island is too small, players search the entire island immediately (in just one period). However, players will search a bigger island in two periods which makes such a search more efficient and more profitable for the players.

Our paper is related to the individual search literature; see Ross (1983) and Gittins (1989). However, players are assumed to search strategically in our model.

Chatterjee and Evans (2004) analyze R&D model which is similar to ours. They allow each of two firms to observe the other’s past choices and search strategically. Their firms have to choose between two research projects. We have only one research project in their notation. Their model is complementary to our model. While agents in their model decide which area to search in (the size is predetermined), agents in our model decide how much area to search (the location has no importance).

Our model has the following main assumptions. First,  $n \geq 2$  players (firms) are searching for a treasure. Second, the treasure (patent/vaccine/prize) value is commonly known. Finally, the total search cost is unknown ex ante. There are

many examples of this situation: detectives (police units) are looking for a criminal; journalists are looking for a movie star in the city hotels; researchers are looking for solutions of the six Millennium Prize Problems in mathematics.

Another striking example is malaria. Malaria is one of the most common infectious diseases and enormous public health problem. Malaria causes about 400 - 900 million cases of fever and approximately one to three million deaths annually - this represents at least one death every 30 seconds.<sup>6</sup> Malaria is not just a disease commonly associated with poverty, but is also a cause of poverty and a major hindrance to economic development. In countries where malaria is common, average per capita GDP has risen (between 1965 and 1990) only 0.4% per year, compared to 2.4% per year in other countries.<sup>7</sup> The economic impact of malaria has been estimated to cost Africa 12 USD billion USD every year. The economic impact includes costs of health care, working days lost due to sickness, days lost in education, decreased productivity, and loss of investment and tourism.<sup>8</sup>

There is currently no vaccine that will prevent malaria (the search costs are unknown *ex ante*), but this is an active field of research. Economic adviser Jeffrey Sachs estimates that malaria can be controlled for US3 billion in aid per year.<sup>9</sup> Therefore, the expected value of a malaria vaccine is at least US3–12 billion per year. Our results show that more firms (more research units) should (in expectations) find a vaccine faster. However, a search with more firms is less efficient (less profitable for the participants) and therefore firms might try to lobby for less “more efficient/productive” units.

The paper is organized as follows. The Model is presented in Section 2. An illustrative example is described in detail in Section 3. A general procedure how to construct a SPME is described in section 4. We discuss properties of the SPME in

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<sup>6</sup>See Breman (2001).

<sup>7</sup>See Sachs, J. and P., Malaney (2002).

<sup>8</sup>See Greenwood, B., Bojang, K., Whitty., C., and G. Targett (2005).

<sup>9</sup>It has been argued that, in order to meet the Millennium Development Goals, money should be redirected from HIV/AIDS treatment to malaria prevention, which for the same amount of money would provide greater benefit to African economies.

Section 5. Section 6 concludes.

## 2 The Model

Suppose that  $n \geq 1$  players are searching for a treasure which is hidden somewhere on an island. The value of the treasure is  $R > 0$  for all players. Let  $x(0) > 0$  and  $x(t) \geq 0$  denote the initial and the current island size. The treasure has equal chances to be at any part of the island.<sup>10</sup> At period  $t \geq 0$ , player  $i$  ( $i = 1, 2, \dots, n$ ) knows the *history*  $h(t) = (x(0); I(0), \dots, I(t-1))$  (where  $I(k) = (I_1(k), \dots, I_n(k)), k = 0, \dots, t-1$ ) and chooses how much to search  $I_i(t)$  (where  $0 \leq I_i(t) \leq x(t)$ ). We assume that the search cost is linear.

If  $I_1(t) + \dots + I_n(t) > x(t)$ , the game ends and each player  $i$  gets payoff

$$-(I_i(0) + \delta I_i(1) + \dots + \delta^t I_i(t)),$$

where  $\delta$  is the common discount factor.<sup>11</sup> Note that we assume (like in the Nash Demand game) that if players decide to search together more than the remaining unexplored island, the treasure is destroyed, the players have to pay the search costs, and the game ends.

If  $I_1(t) + \dots + I_n(t) = x(t)$ , the game ends and each player  $i$  has a  $\frac{I_i(t)}{x(t)}$  chance to find the treasure. The expected payoff of player  $i$  in this case is

$$\delta^t \frac{I_i(t)}{x(t)} R - (I_i(0) + \delta I_i(1) + \dots + \delta^t I_i(t)).$$

If  $I_1(t) + \dots + I_n(t) < x(t)$ , each player  $i$  has a  $\frac{I_i(t)}{x(t)}$  chance to find the treasure and the game ends with probability  $\frac{I_1(t) + \dots + I_n(t)}{x(t)}$ . If the treasure is not found in period  $t$  (this happens with probability  $1 - \frac{I_1(t) + \dots + I_n(t)}{x(t)}$ ), the island size shrinks into  $x(t+1) = x(t) - (I_1(t) + \dots + I_n(t))$ . New size of the island is equal to the previous

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<sup>10</sup>We focus our attention on uniform distribution because this is the most realistic situation when there is no information about the island.

<sup>11</sup>One possible motivation for a discount factor is that there is a  $1 - \delta$  chance that the game terminates at the end of each period.

island size minus the part that has been searched.

We assume that a player can see how much the other players have searched so far before making searching plans for the next period. Note that all investments are sunk, but only one player (if any) can find the treasure.

Player  $i$ 's strategy is an infinite sequence of functions specifying an investment at each period contingent upon any possible sequence of previous investments. Thus, the game we consider is stochastic and any history can be “summarized” by the “state” - the current size of the island. The current size of the island follows a Markov process; that is the probability distribution on the next period state is determined by the current state and the current investments. We will restrict our attention only to Markov strategies in which the past influences the current play only through its effect on the current island size. A pure Markov strategy for player  $i$  is a time-invariant map  $I_i : X \rightarrow X$ , where  $X \in [0, x(0)]$ . Therefore, the solution concept is an efficient *symmetric Markov perfect equilibrium* (SMPE). The total expected payoff is the highest in the efficient SMPE among all SMPE.

Player  $i$  has to solve the following Bellman equation:

$$V(x) = \max_{0 \leq I_i \leq x - I_{-i}} \left( -I_i + \frac{I_i}{x} R + \delta \left( 1 - \frac{I_1 + \dots + I_n}{x} \right) V(x - (I_1 + \dots + I_n)) \right), \quad (1)$$

where  $I_{-i} = I_1 + \dots + I_{i-1} + I_{i+1} + \dots + I_n$ ,  $V(x)$  is each player value function (we use the symmetry assumption here). The first term in equation (1) describes player's costs of investment. The second term is player's expected value from finding the treasure in the current period. The last term is player's expected value from the future periods. Denote the total value of this  $n$ -player game as

$$W_n(x) = nV(x).$$

First, we derive a SMPE for a parametric example in the next section. Then, we investigate general properties of the SMPE.

### 3 Example

Suppose that the value of the treasure is  $R = 1$ , the discount factor is  $\delta = 0.25$ , and the initial island size is  $x(0) = 1$ . In this section, we consider  $n = 3$  players and answer the following questions: What is the SMPE? What is the maximal number of periods (the worst case scenario) when the players find the treasure for sure in the SMPE?

We use the value-iteration method to derive SMPE. To make the exposition clear, as a start, let us assume that the players can search only once. Denote the value function of each player in this case by  $V_1(x)$ . How much should each player search in a SMPE? Note that player 1's expected value from the search if he is allowed to search only once is

$$V_1(x) = \max_{0 \leq I_1 \leq x - I_2 - I_3} -I_1 + R \frac{I_1}{x} = \left( \frac{R}{x} - 1 \right) I_1.$$

It is evident that each player wants to search as much as possible, if  $R \geq x$ .<sup>12</sup> Since in the example,  $R = x$ , each player searches  $I_1 = I_2 = I_3 = x(0)/3$  in the SMPE. Therefore,

$$V_1(x) = \frac{R - x}{3} = \frac{1 - x}{3}. \quad (2)$$

In particular,  $V_1(1) = 0$ . Now, suppose that the players can search for at most two periods. Denote the value function of each player in this case by  $V_2(x)$ . How much should each player search in the first period (if at all) and in the second period in a SMPE? Player 1's expected value from the search if he is allowed to search for at most two periods is

$$V_2(x) = \max_{0 \leq I_1 \leq x - I_2 - I_3} \left( R \frac{I_1}{x} - I_1 \right) + \delta \left( 1 - \frac{I_1 + I_2 + I_3}{x} \right) V_1(x - I_1 - I_2 - I_3),$$

where the first bracket is the expected value of finding the treasure in the first period

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<sup>12</sup>When  $x=R$ , the players are indifferent between searching and not searching. For simplicity we assume that whenever players are indifferent they choose to search.

and the second term is the expected value of finding the treasure in the second period. Note that if the treasure is not found in the first period, the unexplored island shrinks to  $(x - I_1 - I_2 - I_3)$  in the second period and player 1 obtains the expected value  $V_1(x - I_1 - I_2 - I_3)$  which is described in (2).

Since the second-period island  $(x - I_1 - I_2 - I_3)$  is never greater than the first-period island  $x$  and  $R \geq x$  in the example, using (2), we get

$$V_2(x) = \max_{0 \leq I_1 \leq x - I_2 - I_3} \left( R \frac{I_1}{x} - I_1 \right) + \delta \left( 1 - \frac{I_1 + I_2 + I_3}{x} \right) \frac{R - x + I_1 + I_2 + I_3}{3}.$$

The optimal search in the first period,  $I_1$ , satisfies the first order condition

$$\left( \frac{R}{x} - 1 \right) + \delta \left( -\frac{1}{x} \right) \frac{R - x + I_1 + I_2 + I_3}{3} + \delta \left( 1 - \frac{I_1 + I_2 + I_3}{x} \right) \frac{1}{3} = 0,$$

or

$$3(R - x) - \delta(R - x + I_1 + I_2 + I_3) + \delta(x - I_1 - I_2 - I_3) = 0.$$

In the symmetric equilibrium  $I_1 = I_2 = I_3$  and  $I_1 + I_2 + I_3 \leq x(0)$ . Therefore, the previous equation becomes

$$3(R - x) - \delta(R - x + 3I_1) + \delta(x - 3I_1) = 0.$$

Therefore,

$$I_1 = I_2 = I_3 = \frac{3(R - x) - \delta(R - 2x)}{6\delta} = 2(1 - x) - \frac{1}{6}(1 - 2x)$$

and

$$I_1 + I_2 + I_3 = 6(1 - x) - 0.5(1 - 2x) \leq x.$$

Hence, we obtain

$$I_1 = I_2 = I_3 = \begin{cases} \frac{11-10x}{6}, & \text{if } \frac{11}{12} < x \leq 1, \\ x/3, & \text{if } x \leq \frac{11}{12}. \end{cases} \quad (3)$$

Expression (3) means that if the island is relatively big, the first inequality holds, then it is optimal to search for (at most) two periods. However, if the island is relatively small, the second inequality holds, then it is optimal to search the whole island in just one period.

In particular, if  $x(0) = 1$ ,

$$I_1 = I_2 = I_3 = \frac{1}{6}.$$

Therefore, the value function on the interval  $[0, 1]$  is

$$V_2(x) = \begin{cases} \frac{1}{x} \left\{ (1-x) \left( \frac{11-10x}{6} \right) + \frac{1}{12} \left( \frac{12x-11}{2} \right) \left( \frac{13-12x}{2} \right) \right\}, & \text{if } \frac{11}{12} < x \leq 1, \\ V_1(x) = \frac{1-x}{3}, & \text{if } x \leq \frac{11}{12}. \end{cases}$$

or

$$V_2(x) = \begin{cases} \frac{1}{48x} \{-64x^2 + 120x - 55\}, & \text{if } \frac{11}{12} < x \leq 1, \\ V_1(x) = \frac{1-x}{3}, & \text{if } x \leq \frac{11}{12}, \end{cases} \quad (4)$$

where point  $x = \frac{11}{12}$  is the knot of the value function. The value function is described by two different functions on the left and on the right of the knot, but both these functions have the same value on the knot.

In particular,

$$V_2(1) = \frac{1}{48}.$$

The total value of this 3-player game if all players can search at most two times is

$$W_3(x) = 3V_2(x).$$

Figure 1 illustrates the total value in this case.

Now, suppose that the players can search for at most three periods. Denote the value function of each player in this case by  $V_3(x)$ . How much should each player search in the first period (if at all), in the second period, and in the third period in a SMPE? Player 1's expected value from the search if he is allowed to search for at

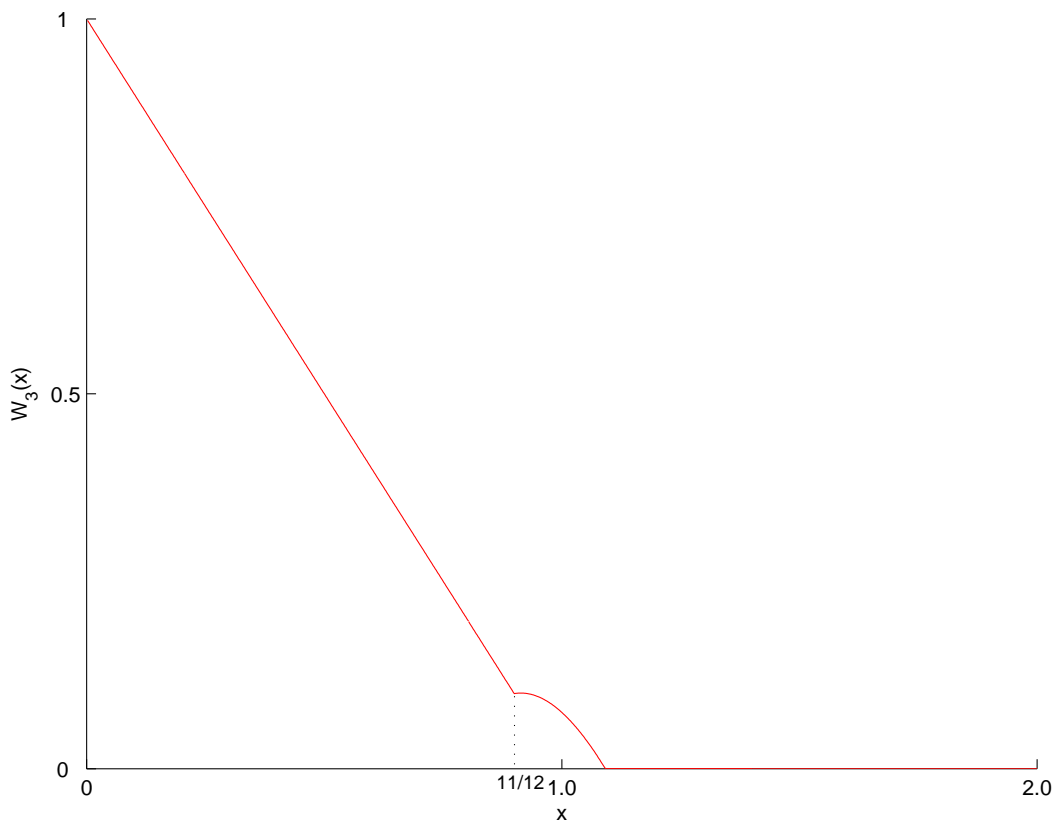


Figure 1:  $W_3(x) = 3V_2(x)$ .

most three periods is

$$V_3(x) = \left( R \frac{I_1}{x} - I_1 \right) + \delta \left( 1 - \frac{I_1 + I_2 + I_3}{x} \right) V_2(x - I_1 - I_2 - I_3), \quad (5)$$

where the first bracket is the expected value of finding the treasure in the first period and the second term is the expected value of finding the treasure after the first period. Note that if the treasure is not found in the first period, the unexplored island shrinks to  $(x - I_1 - I_2 - I_3)$  in the second period and each player obtains the expected value  $V_2(x - I_1 - I_2 - I_3)$  which is described in (4).

Suppose that in the equilibrium,  $x(0) - I_1 - I_2 - I_3 > \frac{11}{12}$ . Then using (4), we get

$$V_3(x) = \left( \frac{I_1}{x} - I_1 \right) + \frac{\delta}{48x} \left\{ -64(x - I_1 - I_2 - I_3)^2 + 120(x - I_1 - I_2 - I_3) - 55 \right\}.$$

The optimal search in the first period,  $I_1$ , satisfies the first order condition

$$\left( \frac{1}{x} - 1 \right) + \frac{\delta}{48x} \{ 128(x - I_1 - I_2 - I_3) - 120 \} = 0,$$

or

$$48 - 48x - \delta(120 - 128x + 128I_1 + 128I_2 + 128I_3) = 0.$$

In the symmetric equilibrium  $I_1 = I_2 = I_3$  and  $x(0) - I_1 - I_2 - I_3 > \frac{11}{12}$ . Therefore,

$$I_1 = I_2 = I_3 = \frac{1}{384\delta} (48 - 120\delta - x(48 - 128\delta)).$$

In particular, for  $x(0) = 1$  and  $\delta = 0.25$ , we get

$$I_1 = I_2 = I_3 = \frac{1}{48}.$$

Since  $x(0) - I_1 - I_2 - I_3 = \frac{15}{16} > \frac{11}{12}$ , which means the three period path for  $x(0) = 1$  exists. However, if we compare expressions (4) and (5) we can notice that  $V_3(x) < V_2(x) \forall x \leq 1$ . That implies that  $V_2(x)$  is the SMPE of this game.

Hence, each player should search  $1/6$  in the first period and, if the treasure is not

found, another  $1/6$  in the second period in the SMPE. It is always optimal to search for at most (in the worst case scenario) two periods even if players can search for *any* number of periods. The two-period search procedure we have described is a SMPE of the game. This example illustrates our approach which we will apply in the general model. It also demonstrates the optimal search delay.

### 3.1 Puzzle 1

Note that from (4),

$$V(0.9167) \approx 0.0278 < 0.0279 \approx V(0.927). \quad (6)$$

It means that a bigger island could make all players better off. This puzzling observation comes from players' inefficient behavior. If players should search the whole island in just one period in the SMPE, players search too fast and it is inefficient. If the size of the island becomes bigger, players search the island slower (more efficiently) in two periods.

### 3.2 Puzzle 2

Using the same methodology one can solve the same problem for any  $n \geq 1$  players. One can show that for any  $n \geq 1$  and  $x \leq 1$  in SMPE players invest twice. When the prize is equal to the island size, i.e.  $x = 1$ , the combined investment in the first and the second period is independent of number of players. In particular, when  $\delta = 0.25$  both combined investments are equal to  $1/2$ .

## 4 Analysis of the Model: SPME

Define the part of the island which player  $i$  does not search in the current period by

$$y := x - I_i$$

and the part of the island which all  $n$  players do not search in the current period (the remaining part of the island) by

$$z := x - (I_1 + \dots + I_n) = x - (I_i + I_{-i}).$$

Equation (1) can be rewritten in the following way

$$V(x) = \max_{I_{-i} \leq y \leq x} \{-(x-y) + R(x-y)/x + \delta zV(z)/x\}. \quad (7)$$

Note,  $x, y, z, R$  and  $V(x)$  are of the same unit measure. For convenience, we make the following substitution

$$x := x/R, \quad y := y/R, \quad z := z/R, \quad V := V/R \quad (8)$$

to work with unit free variables. Equation (7) transforms into

$$V(x) = \max_{I_{-i} \leq y \leq x} \{-(x-y) + (x-y)/x + \delta zV(z)/x\}. \quad (9)$$

Let us derive player's value of the game,  $V(x)$ . To simplify the exposition it is convenient to introduce the following function

$$\Psi(x) := xV(x). \quad (10)$$

From definition (10) it follows that

$$\Psi(x) \geq 0 \text{ for any } x. \quad (11)$$

Note that in the symmetric equilibrium,  $I_1 = \dots = I_n = I$  and  $I_{-i} = (n-1)I$ . Equation (9) in terms of  $\Psi(x)$  can be rewritten as

$$\Psi(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta \Psi(z)\} =: B\Psi(z). \quad (12)$$

The following Lemma follows from the contraction mapping theorem, see for example Stokey, Lucas and Prescott (1989).

**Lemma 1.** *If  $\delta < 1$ , the operator on the right hand side of equation (12) is a contraction operator. Therefore, equation (12) has a unique solution,  $\Psi$ , that can be obtained as the limit of the following sequence  $\{\Psi_k\}$ , where*

$$\Psi_0 \equiv 0, \quad \Psi_k := B\Psi_{k-1} \quad k = 1, 2, \dots \quad (13)$$

#### 4.1 Construction of sequences $\{\Psi_k\}$ and $\{V_k\}$ .

Note that with the help of Lemma 1 we can construct sequence  $\{\Psi_k\}$ . This procedure, it is called the value-iteration procedure, is equivalent to using the backward induction argument and had already been discussed in the Example.

##### 4.1.1 Construction of $\Psi_1$ and $V_1$ .

Let us start from the end of the search process. What will be the value of the game, if players could only search at most one period? Equation (12) transforms into

$$\Psi_1(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y)\}. \quad (14)$$

It is evident that the optimal  $y$  can be described in the following way<sup>13</sup>

$$y = \begin{cases} x, & \text{if } x > 1, \\ (n-1)I, & \text{if } x \leq 1. \end{cases} \quad (15)$$

If  $x < 1$ , then in the SMPE players search the whole island,  $I_1 + \dots + I_n = nI = x$ . Consequently,

$$y = \frac{(n-1)x}{n}, \quad \text{and } z = 0, \quad \text{if } x < 1. \quad (16)$$

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<sup>13</sup>Note that if  $x = 1$ , then any  $y \in [(n-1)I, x]$  is optimal. We assume that players choose  $y = (n-1)I$  in this case.

Therefore, (14) can be rewritten in the following way

$$\Psi_1(x) = \begin{cases} P_1(x), & \text{if } x \leq u_1 = 1, \\ 0, & \text{if } x > u_1 = 1, \end{cases} \quad (17)$$

where  $u_1 > 0$  is a maximal positive root of polynomial  $P_1(x) = x(1-x)/n$ . It is evident that  $u_1 = 1$ . For the future references note that

$$P_1(x) = \frac{a_1}{n}(1-x)^2 + \frac{b_1}{n}(1-x) + \frac{c_1}{n}, \quad (18)$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0.$$

If the players can search the island in at most one period, then the SMPE is  $(I(x), \dots, I(x))$ , where

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if } x \leq u_1, \\ 0, & \text{if } x > u_1. \end{cases} \quad (19)$$

The optimal first-period search is independent from the discount factor because there is no delay.

Define the value of the game for each player (if the players can search the island in at most  $k$  periods), as  $V_k(x) := \Psi_k(x)/x$ , for any  $x \geq 0$ . The value of the game  $V$  can be obtained as the limit of the sequence  $\{V_k\}$ . From the above definition it follows

$$V_1(x) = \begin{cases} (1-x)/n, & \text{if } x \leq u_1, \\ 0, & \text{if } x > u_1. \end{cases} \quad (20)$$

#### 4.1.2 Construction of $\Psi_2$ and $V_2$ .

What will be the value of the game, if players can search the whole island in at most two periods? In general there could be three possibilities depending on the island size. The first possibility is that the players search the whole island in just one period. Intuitively this happens for small values of  $x$  because it is too costly to wait

for another period when the island is very small. The second possibility is that the players finish the island in two periods. This happens for middle values of  $x$ . Finally, players can find searching to be too costly and don't search at all. This happens when the initial island is too big (costs are very large).

We have already considered the first possibility in the previous subsection. Now we formally analyze the situation when players search for exactly two periods in the worst case. The first step is to construct  $\Psi_2(x)$ . Equation (12) in this case transforms into

$$\Psi_2(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi_1(z)\}. \quad (21)$$

Necessary conditions for  $y$  to be the optimal value in the interior of  $[0, x]$  is

$$-(1-x) + \delta\Psi_1'(z) = 0 \quad (22)$$

and

$$0 < z \leq u_1. \quad (23)$$

The sufficient condition for  $y$  to be the optimal value in the interior of  $[0, x]$  is satisfied because

$$\Psi_1''(z) = a_1 < 0. \quad (24)$$

From condition (22) and expression (17), it follows

$$-(1-x) + \delta \left( \frac{1-2z}{n} \right) = 0.$$

Consequently,

$$z(x) = \frac{n(x-1) + \delta}{2\delta}. \quad (25)$$

It is straightforward now that

$$y = \frac{x(n-1) + z(x)}{n} = \frac{2\delta(n-1)x + n(x-1) + \delta}{2\delta n}. \quad (26)$$

Plugging (26) into equation (21) and using (11) we obtain a *spline* of degree two on

the interval  $[0, u_2]$

$$\Psi_2(x) = \begin{cases} \Psi_1(x), & \text{if } 0 \leq x \leq t_1, \\ P_2(x), & \text{if } t_1 < x \leq u_2, \\ 0, & \text{if } x > u_2, \end{cases} \quad (27)$$

where  $u_2 > 0$  is a maximal positive root of polynomial

$$P_2(x) = \frac{a_2}{n}(1-x)^2 + \frac{b_2}{n}(1-x) + \frac{c_2}{n}, \quad (28)$$

where

$$a_2 = -1 - s, \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{\delta}{4},$$

and

$$s = \frac{n(n-2)}{4\delta}. \quad (29)$$

Note that all our calculations so far are valid for any  $n \geq 1$ . Consider parameter  $s$  now. From expression (29), we get that

$$s \begin{cases} < 0, & \text{if } n = 1, \\ = 0, & \text{if } n = 2, \\ > 0, & \text{if } n \geq 3. \end{cases} \quad (30)$$

Condition (30) characterizes three different types of behavior in SMPE. There are three cases:  $n = 1$  (a monopolist);  $n = 2$  (two players); and  $n \geq 3$  (many players).

It is easy to check that

$$u_2 = 1 + \frac{\sqrt{4\delta(s+1)+1}-1}{4(s+1)}.$$

Point  $x = t_1$  is the knot of the spline. The knot  $t_1$  is an initial island size such that players are indifferent between searching the island in two periods or in one period:

$$\Psi_1(t_1) = \Psi_2(t_1). \quad (31)$$

From (18) and (28) we find<sup>14</sup>

$$t_1 = 1 - \frac{\delta}{n}. \quad (32)$$

It is straightforward to check that conditions (23) hold for any initial island size  $x$  in the expression (27). Therefore, if the players can search the island in at most two periods, then  $y(x)$  is a spline of degree one on the interval  $[0, u_2]$  with knot  $x = t_1$

$$y(x) = \begin{cases} \frac{(n-1)x}{n}, & \text{if } x \leq t_1, \\ \frac{2\delta(n-1)x + n(x-1) + \delta}{2\delta n}, & \text{if } t_1 < x \leq u_2, \\ x, & \text{if } x > u_2, \end{cases} \quad (33)$$

the SMPE (if the players can search the island in at most 2 periods) is also a spline of degree one on the interval  $[0, u_2]$  with knot  $x = t_1$

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if } x \leq t_1, \\ \frac{(2\delta-n)x + n - \delta}{2\delta n}, & \text{if } t_1 < x \leq u_2, \\ 0, & \text{if } x > u_2, \end{cases} \quad (34)$$

and the value function is

$$V_2(x) = \begin{cases} V_1(x), & \text{if } x \leq t_1, \\ P_2(x)/x, & \text{if } t_1 < x \leq u_2, \\ 0, & \text{if } x > u_2. \end{cases} \quad (35)$$

#### 4.1.3 Puzzle 1.

Now, we look at puzzle 1 which we discovered in the Example. It looks intuitive that if the treasure is hidden on a smaller island, all players should be better off in the SPME. It seems that costs go down (smaller island), but the probability to find the treasure does not change in SPME. However, as we saw in the Example, this intuition does not take into account that players search too fast (over-search) a small island. It turns out that sometimes the last effect can be so strong that all players are better

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<sup>14</sup>Note that condition  $t_1 \leq u_1 = 1$  must hold.

off searching bigger island. The following result demonstrates that this puzzle can take place for at least  $n = 3$  players.

**Proposition 1.** *For any  $n \geq 3$ , there exists  $x > t_1$  such that  $V(x) > V(t_1)$ .*

**Proof.** See the Appendix.

#### 4.1.4 Construction of $\Psi_k$ and $V_k$ .

What will be the value of the game, if players can search the whole island in at most  $k \geq 3$  periods? In general there could be  $k + 1$  possibilities depending on the island size. The players can search the island in 1, 2, ...,  $k$  periods or don't search at all.

First, we construct  $\Psi_k(x)$ . Equation (12) in this case transforms into

$$\Psi_k(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi_{k-1}(z)\} =: B\Psi_{k-1}(z). \quad (36)$$

A necessary condition for  $y$  to be the optimal value in the interior of  $[0, x]$  is

$$(1-x) = \delta\Psi'_{k-1}(z). \quad (37)$$

and

$$t_{k-2} < z \leq u_{k-1}, \quad (38)$$

where  $t_0 = 0$ . The sufficient condition for  $y$  to be the optimal value in the interior of  $[0, x]$  is satisfied if

$$\Psi''_{k-1}(z) < 0. \quad (39)$$

Note that if function  $\Psi_{k-1}(x)$  in (36) is a quadratic polynomial,  $\Psi_k(x) = B\Psi_{k-1}(x)$  has to be a quadratic polynomial as well. Since from (18) and (28),  $P_1(x)$  and  $P_2(x)$  are quadratic polynomials, any  $P_k(x)$  can be represented in the following form:

$$P_k(x) = \frac{a_k}{n}(1-x)^2 + \frac{b_k}{n}(1-x) + \frac{c_k}{n}, \quad k \geq 1. \quad (40)$$

From condition (37) and expression (40), it follows

$$z(x) = 1 + \frac{\delta b_{k-1} + (1-x)n}{2\delta a_{k-1}}. \quad (41)$$

It is straightforward now that

$$y = x + \frac{z-x}{n} = x + \frac{(1-x)(n + 2\delta a_{k-1})}{2\delta n a_{k-1}} + \frac{b_{k-1}}{2n a_{k-1}}. \quad (42)$$

Hence,

$$\Psi_k(x) = -(1-x) \left( \frac{(1-x)(n + 2a_{k-1}\delta)}{2na_{k-1}\delta} + \frac{b_{k-1}}{2na_{k-1}} \right) + \delta \Psi_{k-1}(z). \quad (43)$$

Define the largest root of polynomial  $P_k(x)$  as  $u_k$  and an initial island size such that players are indifferent between searching the island in  $k$  periods or in  $k-1$  periods as knot  $t_{k-1}$ ,

$$\Psi_{k-1}(t_{k-1}) = \Psi_k(t_{k-1}). \quad (44)$$

Plugging (42) into equation (36) and using (11), we obtain a spline of degree two on the interval  $[0, u_k]$  with knots  $t_1, \dots, t_{k-1}$

$$\Psi_k(x) = \begin{cases} \Psi_{k-1}(x), & \text{if } 0 \leq x \leq t_{k-1}, \\ P_k(x), & \text{if } t_{k-1} < x \leq u_k, \\ 0, & \text{if } x > u_k, \end{cases} \quad (45)$$

where  $P_k(x)$  is defined in (40). Our description will be complete if conditions (38) and (39) hold. The following proposition takes care of that.

**Proposition 2.** *Conditions (38) and (39) hold for any  $n \geq 1$  and any initial island size  $x$  in the expression (45).*

## 4.2 Preliminary results

Let us find  $a_k$ ,  $b_k$ , and  $c_k$  for any  $k \geq 2$  now. Using (36), (40), and (41), we get the following result.

**Theorem 1.**

$$a_k = -1 + \frac{s}{a_{k-1}}, \quad b_k = -\frac{b_{k-1}}{2a_{k-1}}, \quad c_k = \delta \left( c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}} \right), \quad k \geq 2, \quad (46)$$

where  $s$  is defined in (29) and

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0.$$

Theorem 1 describes all coefficients of the quadratic polynomials  $P_k(x)$ . In particular, coefficient  $a_k$  depends on parameter  $s$ . Condition (30) helps us to identify three cases:  $n = 1$  (a monopolist);  $n = 2$  (two players); and  $n \geq 3$  (many players).

### 4.2.1 $n = 1$

We start from the monopolist case. Then, Theorem 1 helps to obtain the following results which characterize the splines in (45) and their knots for  $n = 1$ .

Recall that at the knot  $t_{k-1}$  players are indifferent between searching for  $k$  or  $k - 1$  periods and  $u_k$  is the largest initial island size which is searched in the SMPE (if players can search for at most  $k$  periods).

**Proposition 3.** *Suppose that  $n = 1$ . Then,*

$$a_k = -\frac{\sin(k+1)\varphi}{2v \sin k\varphi}, \quad b_k = \frac{v^{k-1} \sin \varphi}{\sin k\varphi}, \quad c_k = \frac{v^{2k-1} \sin(k-1)\varphi}{2 \sin k\varphi}, \quad (47)$$

$$t_k = 1 - v^k \cos k\varphi, \quad u_k = 1 + \frac{v^k (\sin k\varphi - \sin \varphi)}{\sin(k+1)\varphi}, \quad k \geq 1, \quad (48)$$

where  $v = \sqrt{\delta}$  and  $\varphi = \arccos v$ .

**Proof.** See the Appendix.

### 4.2.2 $n = 2$

We characterize the splines in (45) and their knots for  $n = 2$  here. Since  $s = 0$ , we get the following results.

**Proposition 4.** *Suppose that  $n = 2$ . Then,*

$$a_k = -1, \quad b_k = \frac{1}{2^{k-1}}, \quad c_k = \left( \frac{(4\delta)^{k-1} - 1}{4^{k-1}(4\delta - 1)} \right) \delta, \quad (49)$$

$$t_k = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}, \quad u_k = 1 + \frac{1}{2^k} \left( \sqrt{\frac{1 - (4\delta)^k}{1 - 4\delta}} - 1 \right), \quad k \geq 1. \quad (50)$$

**Proof.** See the Appendix.

### 4.2.3 $n \geq 3$

Finally, we consider  $n \geq 3$  case. The following proposition describes the splines in (45) and their knots in this case.

**Proposition 5.** *Suppose that  $n \geq 3$ . Then,*

$$a_k = \frac{\left( \frac{\sqrt{1+4s-1}}{2} \right)^{k+1} - \left( \frac{-\sqrt{1+4s-1}}{2} \right)^{k+1}}{\left( \frac{\sqrt{1+4s-1}}{2} \right)^k - \left( \frac{-\sqrt{1+4s-1}}{2} \right)^k}, \quad b_k = \frac{\left( -\frac{1}{2} \right)^{k-1} \sqrt{1+4s}}{\left( \frac{\sqrt{1+4s-1}}{2} \right)^k - \left( \frac{-\sqrt{1+4s-1}}{2} \right)^k}, \quad (51)$$

$$c_k = \sum_{i=2}^k \frac{-(1+4s)\delta^k}{(4\delta)^{i-1} \left[ \left( \frac{\sqrt{1+4s-1}}{2} \right)^i - \left( \frac{-\sqrt{1+4s-1}}{2} \right)^i \right] \left[ \left( \frac{\sqrt{1+4s-1}}{2} \right)^{i-1} - \left( \frac{-\sqrt{1+4s-1}}{2} \right)^{i-1} \right]},$$

$$t_k = 1 + \frac{b_{k+1} - b_k - \sqrt{(b_{k+1} - b_k)^2 - 4(a_{k+1} - a_k)(c_{k+1} - c_k)}}{2(a_{k+1} - a_k)}, \quad (52)$$

$$u_k = 1 + \frac{b_k - \sqrt{b_k^2 - 4a_k c_k}}{2a_k}. \quad (53)$$

**Proof.** See the Appendix.

### 4.3 SMPE

We can describe the SMPE now. If the players can search the island in at most  $k$  periods, then  $y(x)$  is a spline of degree one on the interval  $[0, u_k]$  with knots  $t_1, \dots, t_{k-1}$

$$y(x) = \begin{cases} \frac{(n-1)x}{n}, & \text{if } x \leq t_1, \\ x + \frac{(1-x)(n+2\delta a_1)}{2\delta n a_1} + \frac{b_1}{2n a_1}, & \text{if } t_1 < x \leq t_2, \\ \vdots \\ x + \frac{(1-x)(n+2\delta a_{k-2})}{2\delta n a_{k-2}} + \frac{b_{k-2}}{2n a_{k-2}}, & \text{if } t_{k-2} \leq x \leq t_{k-1}, \\ x + \frac{(1-x)(n+2\delta a_{k-1})}{2\delta n a_{k-1}} + \frac{b_{k-1}}{2n a_{k-1}}, & \text{if } t_{k-1} < x \leq u_k, \\ x, & \text{if } x > u_k, \end{cases} \quad (54)$$

the SMPE (if the players can search the island in at most  $k$  periods) is also a spline of degree one on the interval  $[0, u_k]$  with knots  $t_1, \dots, t_{k-1}$

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if } x \leq t_1, \\ -\frac{(1-x)(n+2\delta a_1)}{2\delta n a_1} - \frac{b_1}{2n a_1}, & \text{if } t_1 < x \leq t_2, \\ \vdots \\ -\frac{(1-x)(n+2\delta a_{k-2})}{2\delta n a_{k-2}} - \frac{b_{k-2}}{2n a_{k-2}}, & \text{if } t_{k-2} \leq x \leq t_{k-1}, \\ -\frac{(1-x)(n+2\delta a_{k-1})}{2\delta n a_{k-1}} - \frac{b_{k-1}}{2n a_{k-1}}, & \text{if } t_{k-1} < x \leq u_k, \\ 0, & \text{if } x > u_k, \end{cases} \quad (55)$$

and the value function (if the players can search the island in at most  $k$  periods) is

$$V_k(x) = \begin{cases} P_1(x)/x, & \text{if } x \leq t_1, \\ P_2(x)/x, & \text{if } t_1 < x \leq t_2, \\ \vdots \\ P_{k-1}(x)/x, & \text{if } t_{k-2} \leq x \leq t_{k-1}, \\ P_k(x)/x, & \text{if } t_{k-1} < x \leq u_k, \\ 0, & \text{if } x > u_k, \end{cases} \quad (56)$$

or

$$V_k(x) = \begin{cases} V_{k-1}(x), & \text{if } 0 \leq x \leq t_{k-1}, \\ P_k(x)/x, & \text{if } t_{k-1} < x \leq u_k, \\ 0, & \text{if } x > u_k. \end{cases} \quad (57)$$

## 5 Properties of SMPE

The SMPE was described in the previous Section. We look at its properties now.

First, we show that if  $n = 1, 2$ , then players search the island for a *finite* number of periods or not at all.

### 5.1 Maximum number of periods

In this subsection, we fix the number of players,  $n$ , and ask the following question: what is the minimum number of periods,  $k$ , such that  $V(x) \equiv V_k(x)$ . In other words, what is the maximum number of periods (the worst case scenario) in which the treasure will be found for sure. The answer, of course, depends on  $\delta$ .

Note that from (17) and (32) we get  $u_1 > t_1$ . It means that there exists an island size  $x$  such that, in the worst case scenario, the treasure is found in *at least* two periods. This observation is true for any  $n \geq 1$ .

In general, we have to write a condition when a the biggest positive root of the a quadratic polynomial  $P_k(x)$  coincides with the biggest positive root of the a quadratic polynomial,  $P_{k+1}(x)$ . It means that  $\Psi_k(x) = \Psi_{k+1}(x)$ , or  $V(x) \equiv V_k(x)$ . Since such  $k$  depends on  $\delta$ , we define for each  $n$  a knot discount factor,  $\delta_k(n)$  which is the solution of the following equation

$$u_k(\delta_k(n)) = u_{k+1}(\delta_k(n)) \quad k = 2, 3, \dots \quad (58)$$

The knot discount factor  $\delta_k(n)$  “connects” two regions: if  $0 < \delta < \delta_k(n)$ , then  $V(x) \equiv V_k(x)$  and if  $\delta_k(n) < \delta < 1$ , then  $V_{k+1}(x) \neq V_k(x)$ .

The following theorem characterizes the knot discount factors for  $n = 1$  and 2.

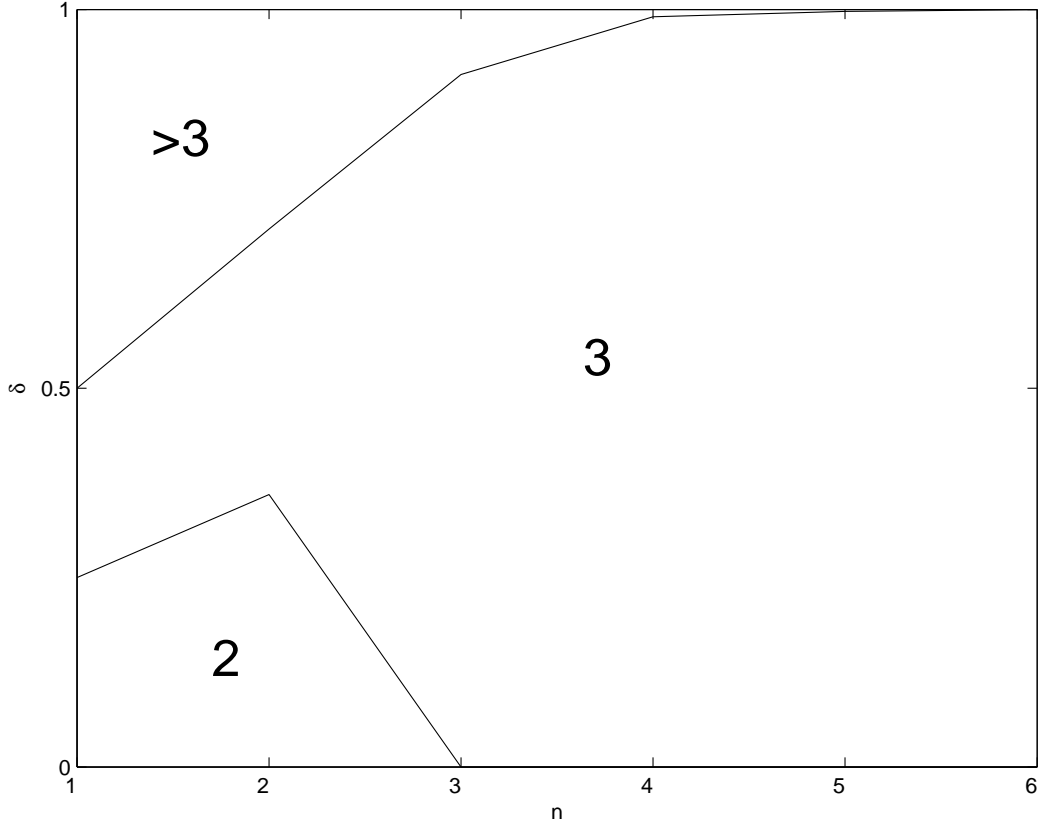


Figure 2: Different zones

**Theorem 2.** *If  $n = 1$ , equation (58) has the following unique solution*

$$\delta_k(1) = \cos^2 \frac{\pi}{k+1}, \quad k = 2, 3, \dots \quad (59)$$

*If  $n = 2$ , equation (58) can be simplified to*

$$(1 - \delta_k(2))^2(1 - (4\delta_k(2))^k) = 1 - 4\delta_k(2), \quad k = 2, 3, \dots \quad (60)$$

**Proof.** See the Appendix.

Figure 2 illustrates the results of this subsection. If  $n = 1$ , we plug in two knot discount factors  $\delta_2(1) = 0.25$  and  $\delta_3(1) = 0.5$ . Numerical calculations for  $n \geq 2$  allow us to draw Figure 2. We can see that if  $n \geq 3$ , the treasure will be found for sure (in the worst case scenario) in *at least* three periods.

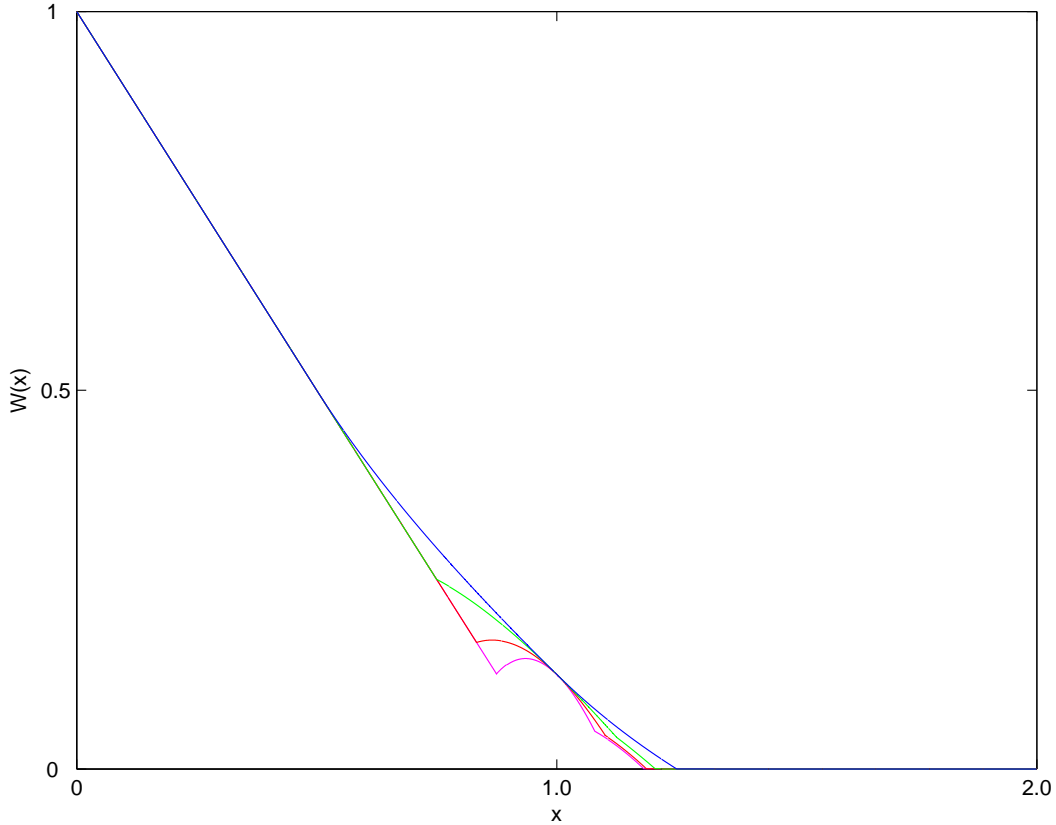


Figure 3:  $W_1(x)$  is a blue line,  $W_2(x)$  is a green line,  $n = 3$  is a red line;  $\delta = 0.5$ .

## 5.2 Puzzle 2

It is intuitive that players will behave as a cartel if the island size is “very small”: players search the whole island in just one period. In this sense, players will play an efficient strategy if the island is “very small”. It is also intuitive that the total value of the game will be higher for smaller  $n$  if the island size is “small.” The monopolist,  $n = 1$ , gets the highest total value, then two players,  $n = 2$ , and so on. However, there is another unique island size,  $x = 1$ , where players play efficient strategy (behave as a cartel). Formally,

**Proposition 6.** *For any  $0 < \delta \leq 0.5$ ,  $W_n(1) = W_1(1)$  for any  $n \geq 2$ .*

Figure 3 illustrates Puzzle 2. It shows the total value function if one, two, three, or four players search the island. The total value function is the same in all four cases for  $x = 1$ .

If the discount factor becomes greater, the monopolist can realize that he does not need to search so fast. There is the following trade off: he can save on the search costs, if the treasure is found in the first period, or he does not lose a lot of the treasure value if the discount factor is high enough. It means that for high discount factors the monopolist will search in two periods, but two, three, and so on firms will still search in just one period.

**Proposition 7.** *For any  $0 < \delta \leq 0.75$ ,  $W_n(1) = W_2(1)$  for any  $n \geq 3$ .*

Figure 4 illustrates Proposition 7. It shows that the total value function of the monopolist,  $n = 1$ , at  $x = 1$  is greater than the total value functions for two, three, or four players at  $x = 1$ . The total value functions are the same at  $x = 1$  for  $n = 2, 3, 4$ .

### 5.3 Puzzle 3: Tragedy of commons or free riding?

Figure 5 illustrates two effects: the tragedy of commons (for  $t_1 < x < 1$ ) and a free riding (for  $x > 1$ ).

### 5.4 Extensions and Applications

There are several natural extensions of our project. The first one is to allow players have a positive externality on each other. For example, all players can benefit from the treasure in some way. The question is how this will affect a SMPE and our results.

It will be very interesting to test our predictions in the experimental laboratory. For example, check whether subjects' behavior is consistent with the SMPE and either Puzzle 1 and 2 hold in experiments.

## 6 Conclusion

In this paper, we analyze a dynamic search model. We suggest a new model and develop new methods in order to analyze our simple setting. Our symmetry assumption makes the analysis simpler and more transparent. We are able to construct the

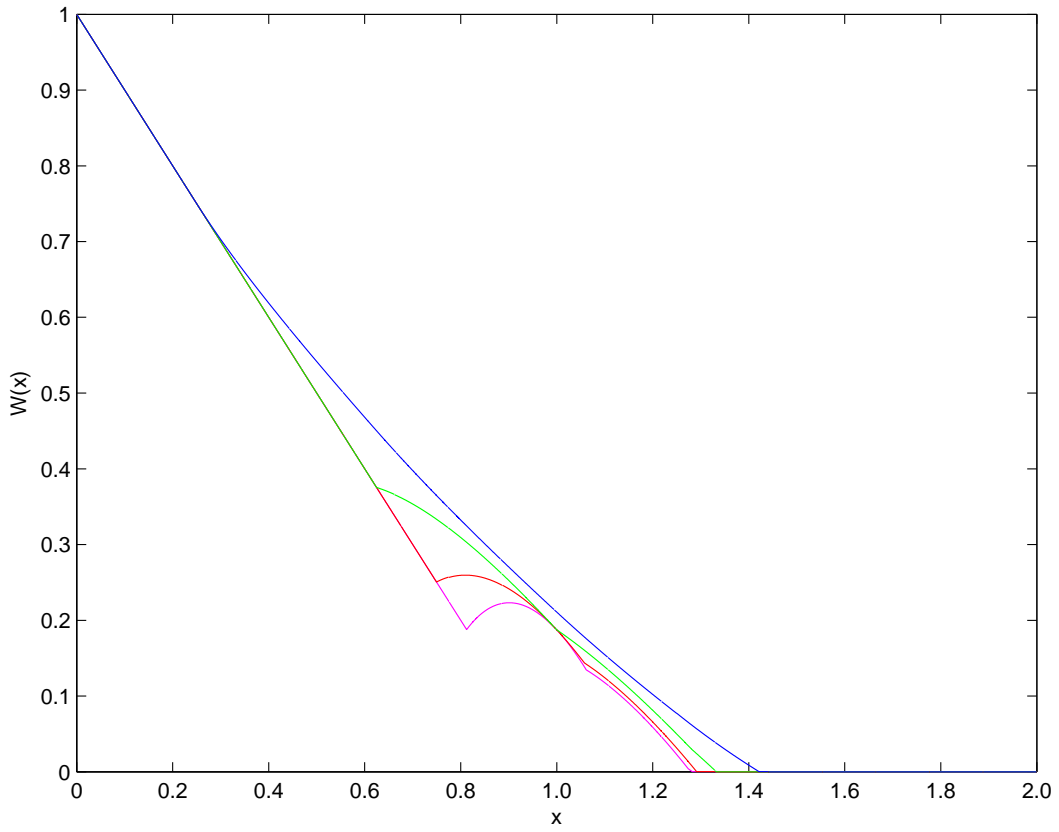


Figure 4:  $W_n(x)$ :  $n = 1$  is a blue line,  $n = 2$  is a green line,  $n = 3$  is a red line;  $\delta = 0.75$ .

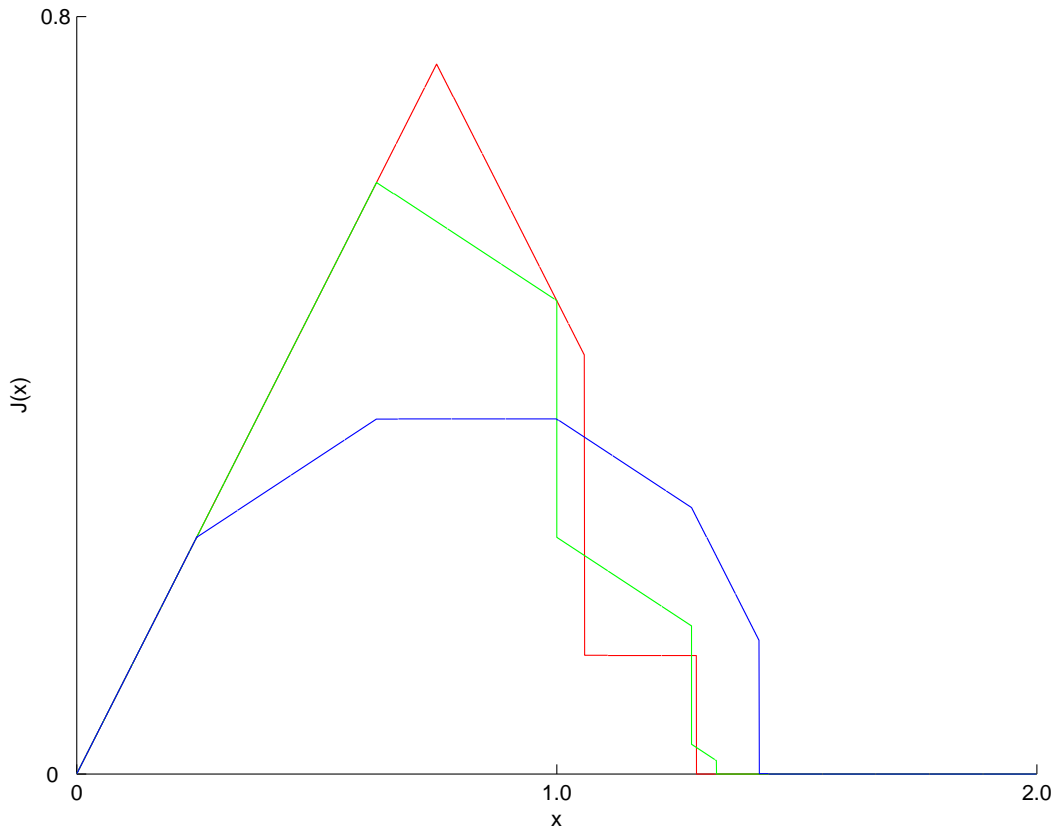


Figure 5:  $J_n(x)$ :  $n = 1$  is a blue line,  $n = 2$  is a green line,  $n = 3$  is a red line;  $\delta = 0.75$ .

efficient SMPE. We demonstrate two types of inefficiencies: a tragedy of commons and free riding.

## Appendix

### Proof of Proposition 1

When  $x = t_1$  players are indifferent between searching the island in two periods or in one period:

$$V_1(t_1) = V_2(t_1).$$

Let us show that  $V_2'(t_1) > 0$  for  $n \geq 3$  which means that there exists an island size

$x$  which is “slightly” bigger than the island size  $t_1$ ,  $x > t_1$ , such that  $V(x) = V_2(x) > V(t_1)$ :

$$V_2'(t_1) = \left( \frac{P_2(x)}{x} \right)'_{x=t_1} > 0. \quad (61)$$

Using (28) and (32), inequality (61) becomes

$$V_2'(t_1) = a_2 - \frac{a_2 + b_2 + c_2}{(1 - \delta/n)^2} > 0. \quad (62)$$

Using (28) and (29), inequality (62) simplifies to:

$$n^2(2n - 6 - 2\delta) + \delta(10n - 4\delta) > 0. \quad (63)$$

When  $n \geq 4$  the above inequality always holds because both terms on the left are positive. When  $n = 3$  inequity (63) simplifies to

$$\delta(12 - 4\delta) > 0,$$

which also holds. The proposition is proved.  $\square$

## Proof of Proposition 2

Firstly, let us show that  $z \leq u_{k-1}$ . Let us prove by contradiction assuming that  $z > u_{k-1}$ . Refer to equation (36) which could be seen below

$$\Psi_k(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi_{k-1}(z)\}.$$

Given  $x \geq y \geq z > u_{k-1} \geq \dots \geq u_2 \geq u_1 = 1$  it follows that the first term on the right  $(1-x)(x-y)$  has to be non-positive. If  $z > u_{k-1}$  then the second term on the right  $\delta\Psi_{k-1}(z)$  is negative. That means the whole expression on the right of equation (36) has to be negative. Obviously that could not be an optimal choice for a player because by choosing  $y = x$ , i.e. making no investment, a player can get

the value of zero. Consequently, we have a contradiction and condition  $z \leq u_{k-1}$  is proved.

Now let us show that  $t_{k-2} \leq z$ . Let us prove by contradiction assuming that  $z < t_{k-2}$ . Note that from construction when  $z < t_{k-2}$ , the following condition has to hold  $P_{k-1}(z) < P_{k-2}(z)$ . That implies that instead of considering the original  $k$ -period path (investment  $x - y$  in the first period and further  $k - 1$  investments according to  $P_{k-1}(z)$ ) we could use  $k - 1$  period path (same investment  $x - y$  in the first period and further  $k - 2$  investments according to  $P_{k-2}(z)$ ) and increase the value. Refer to equation (36), both paths have the same first term while the second term is larger for the  $k - 1$  period path. That implies the  $k$  period path does not improve the value in comparison with the optimal  $k - 1$  period path, which means whenever  $z < t_{k-2}$  the  $k$  period path does not exist. Condition  $t_{k-2} \leq z$  is proved.  $\square$

## Proof of Proposition 3

### Derivation of $a_k$ , $b_k$ and $c_k$

Let us show that when  $n = 1$  formula (47) describes the solution to the system of difference equations (46).

Define

$$P_k := v^k \cdot \prod_{j=1}^k a_j \quad k = 1, 2, \dots \quad (64)$$

Using (46) we get the following second-order difference equation

$$P_{k+1} = vP_k \cdot \left( -1 - \frac{1}{4\delta a_k} \right) = -vP_k - \frac{1}{4}P_{k-1} \quad k \geq 2. \quad (65)$$

The initial conditions are  $P_0 = 1$  and  $P_1 := -v$ . The characteristic equation  $4z^2 + 4vz + 1 = 0$  has two complex roots

$$z_1 = \frac{-v + ir}{2}, \quad z_2 = \frac{-v - ir}{2}, \quad r := \sqrt{1 - v^2} > 0. \quad (66)$$

Denote  $\varphi := \{\arg z_1 \in [0, \pi/2]\} = \arccos v$ , then  $z_{1,2} = -\frac{e^{\pm i\varphi}}{2}$ . Further, write

solutions to equation (65) in form  $P_k = Az_1^{k+1} - Bz_2^{k+1}$  and use initial conditions to get  $A = B = -\frac{i}{\sin \varphi}$ . Consequently

$$P_k = -\frac{i}{(-2)^{k+1} \sin \varphi} (e^{i(k+1)\varphi} - e^{-i(k+1)\varphi}) = -\frac{\sin [(k+1)\varphi]}{(-2)^k \sin \varphi}. \quad (67)$$

Apply (64) and (46) to get

$$a_k = \frac{P_k}{vP_{k-1}} = -\frac{\sin (k+1)\varphi}{2v \sin k\varphi}, \quad (68)$$

$$b_k = -\frac{b_{k-1}}{2a_{k-1}} = \frac{v^{k-1} \sin \varphi}{\sin k\varphi}, \quad (69)$$

$$c_k = \delta \left[ c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}} \right] = \frac{v^{2k-1} \sin (k-1)\varphi}{2 \sin k\varphi}. \quad (70)$$

**Derivation of  $t_k$**

*Using*

$$t_k - 1 = a_{k-1}\delta(t_{k-1} - 1) + b_{k-1}\delta, \quad (71)$$

*and the initial condition  $t_1 = 1 - \delta$  let us show that*

$$t_k = 1 - v^k \cos k\varphi. \quad (72)$$

Substitute (72) into (71) to get

$$(t_k - 1) \sin (k-1)\varphi = v(t_{k-1} - 1) \sin k\varphi + v^k \sin \varphi. \quad (73)$$

Substitute the following equality

$$\sin \varphi = \sin k\varphi \cos (k-1)\varphi - \cos k\varphi \sin (k-1)\varphi \quad (74)$$

into (73) to get

$$(t_k - 1 + v^k \cos k\varphi) \sin(k-1)\varphi = (t_{k-1} - 1 + v^{k-1} \cos(k-1)\varphi)v \sin k\varphi. \quad (75)$$

It is easy to see that (72) is the solution to (75) that satisfies the initial condition  $t_1 = 1 - \delta$ .

### Derivation of $u_k$

Using

$$a_k(1 - u_k)^2 + b_k(1 - u_k) + c_k = 0 \quad (76)$$

let us show that

$$u_k = 1 + \frac{v^k(\sin k\varphi - \sin \varphi)}{\sin(k+1)\varphi}. \quad (77)$$

Substitute (46) into (76) to get

$$-(1 - u_k)^2 \sin(k+1)\varphi + 2(1 - u_k)v^k \sin \varphi + v^{2k} \sin(k-1)\varphi. \quad (78)$$

Solving this quadratic equation results in (77).  $\square$

## Proof of Proposition 4

### Derivation of $a_k$ , $b_k$ and $c_k$

Let us show that when  $n = 2$  formula (49) describes the solution to the system of difference equations (46). It is straightforward to derive  $a_k = -1$  and  $b_k = \frac{1}{2^{k-1}}$ . The expression for  $c_k$  in (46) can be simplified to

$$c_k = \delta(c_{k-1} + 1/4^{k-1}). \quad (79)$$

Introduce new variable  $e_k = c_k 4^k$ . Equation (79) transforms to

$$e_k = 4\delta(e_{k-1} + 1), \quad (80)$$

where  $e_1 = 0$ . The solution to this linear difference equation is  $e_k = \frac{4\delta - (4\delta)^k}{1 - 4\delta}$ . Use  $c_k = e_k/4^k$  to derive

$$c_k = \frac{4\delta - (4\delta)^k}{(1 - 4\delta)4^k}. \quad (81)$$

### Derivation of $t_k$

Applying equations (44) and (40) we get

$$a_k(1 - t_k)^2 + b_k(1 - t_k) + c_k = a_{k+1}(1 - t_k)^2 + b_{k+1}(1 - t_k) + c_{k+1}, \quad k = 1, 2, \quad (82)$$

From equation (49)  $a_k = a_{k+1} = -1$ ; consequently,

$$t_k = 1 + \frac{c_{k+1} - c_k}{b_{k+1} - b_k}. \quad (83)$$

Substituting  $b_k$  and  $c_k$  from equation (49), we derive the following indifference points

$$t_k = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}. \quad (84)$$

### Derivation of $u_k$

Applying equations (44) and (40) we get

$$a_k(1 - u_k)^2 + b_k(1 - u_k) + c_k = 0, \quad k = 1, 2, \quad (85)$$

Substituting  $a_k = -1$  from equation (49) and solving the above quadratic equation gives

$$u_k = 1 + \frac{\sqrt{b_k^2 + 4c_k} - b_k}{2}. \quad (86)$$

Note that applying (49) one can simplify

$$b_k^2 + 4c_k = \frac{(4\delta)^k - 1}{4^{k-1}(4\delta - 1)}. \quad (87)$$

Applying equation (87) to equation (86) one can get

$$u_k = 1 + \frac{\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1}{2^k}. \quad (88)$$

□

## Proof of Proposition 5

### Derivation of $a_k$ , $b_k$ and $c_k$

Using (46) let us show (51).

Define

$$P_k := \prod_{j=1}^k a_j \quad k = 1, 2, \dots \quad (89)$$

Using (46) we get the following second-order difference equation

$$P_k + P_{k-1} - sP_{k-2} = 0 \quad k \geq 2. \quad (90)$$

The initial conditions are  $P_1 = -1$  and  $P_2 := 1 + s$ . The characteristic equation  $z^2 + z - s = 0$  has two real roots

$$z_1 = \frac{\sqrt{1+4s} - 1}{2}, \quad z_2 = \frac{-\sqrt{1+4s} - 1}{2}. \quad (91)$$

The solutions to equation (90) have to be in form  $P_k = Az_1^{k+1} - Bz_2^{k+1}$ . Applying initial conditions we get  $A = B = \frac{1}{\sqrt{1+4s}}$ . Next, from  $a_k = \frac{P_k}{P_{k-1}}$  we derive  $a_k$  in (51).

Note that from (46) it follows that  $a_{k-1} = -\frac{b_{k-1}}{2b_k}$ . On the other hand,  $a_{k-1} = \frac{P_{k-1}}{P_{k-2}}$ . Consequently,  $b_k$  is inversely proportional to  $P_{k-1}$ . Next we find  $b_k$  in (51) by substitution.

$c_k$  has to satisfy the initial condition  $c_1 = 0$ . Introducing  $d_k = \frac{c_k}{\delta^k}$  and rewriting the difference equation gives

$$d_k = d_{k-1} + \frac{b_k b_{k-1}}{2\delta^{k-1}}.$$

Substituting the initial condition  $d_1 = 0$

$$d_k = d_1 + \sum_{i=2}^k (d_i - d_{i-1}) = \sum_{i=2}^k \frac{b_i b_{i-1}}{2\delta^{i-1}}.$$

Finally we have

$$c_k = \delta^k d_k = \delta^k \sum_{i=2}^k \frac{b_i b_{i-1}}{2\delta^{i-1}}.$$

Substituting  $b_k$  from (51) we derive  $c_k$  in (51).

### Derivation of $t_k$ and $u_k$

The players' indifference points are

$$t_k = 1 + \frac{b_{k+1} - b_k - \sqrt{(b_{k+1} - b_k)^2 - 4(a_{k+1} - a_k)(c_{k+1} - c_k)}}{2(a_{k+1} - a_k)}. \quad (92)$$

$\Psi_k(x)$  is strictly positive for any  $0 < x < u_k$  and zero for any  $x \geq u_k$ , where

$$u_k = 1 + \frac{b_k - \sqrt{b_k^2 - 4a_k c_k}}{2a_k}. \quad (93)$$

It follows from equations (44) and (40).  $\square$

## Proof of Theorem 2

Firstly let us prove equation (59). Equation (59) (which defines  $\delta_k$ ) is equivalent to condition  $P(k-1) = 0$  ( $P_k$  is defined in (89)). Apply (67) to get

$$\varphi_k \quad : \quad k\varphi = n\pi, \quad (94)$$

where  $n \geq 1$  is some natural number which in general can depend on  $k$ , i.e.  $n = n_k$ .

Let us prove by induction that  $n_k = 1 \forall k$ . It is easy to see that for  $k = 1$  the statement is correct. Substitute  $k' := k + 1$  in (94) to get (using assumption that  $n_k = 1$  and the fact that  $\varphi_k = \arccos \sqrt{\delta_k}$  is monotonically decreasing in  $k$ )

$$n_{k+1}\pi = (k+1)\varphi_{k+1} \leq (k+1)\varphi_k = \frac{k+1}{k}\pi < 2\pi.$$

Given that  $n_{k+1}$  is a natural number, it has to be the case that  $n_{k+1} = 1$ . Substitute  $n = 1$  in (94) to get  $\varphi_k = \pi/k$ , which means (59) is proved.

Now let us prove equation (60). Substitute expressions for  $t_k$  from equation (50) and expressions for  $u_k$  from equation (88) into equation (58) to get

$$1 + \frac{\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1}{2^k} = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}. \quad (95)$$

Simplify the above expression to

$$\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1 = \frac{3\delta + (4\delta)^k(\delta - 1)}{(4\delta - 1)} \quad (96)$$

Further simplifications give

$$\sqrt{\frac{(4\delta)^k - 1}{4\delta - 1}} = \frac{((4\delta)^k - 1)(1 - \delta)}{(4\delta - 1)} \quad (97)$$

and

$$\sqrt{\frac{(4\delta)^k - 1}{4\delta - 1}}(1 - \delta) = 1. \quad (98)$$

Squaring both sides of equation (98) results in relationship (60), which proves theorem 2.  $\square$

## Proof of Proposition 6

To prove this proposition one needs to show two things. Firstly, one needs to show that  $t_2 \geq 1$  for any  $n \geq 1$ , which implies  $V(1) = V_2(1)$  for any  $n$ . Secondly, one needs to show that  $nV_2(1)$  is the same for any  $n$ .

## Proof of Proposition 7

To prove this proposition one needs to show two things. Firstly, one needs to show that  $t_2 \geq 1$  for any  $n \geq 2$ , which implies  $V(1) = V_2(1)$  for any  $n \geq 2$ . Secondly, one needs to show that  $n * V_2(1)$  is the same for any  $n \geq 2$ .

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