GIFT EQUILIBRIA AND PARETO IMPROVEMENTS

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Abstract

Piecemeal reform methodology is used to identify Pareto improving directions of gifts in the economy studied by L. Kranich, "Gift Equilibria and Pareto Optimality Reconsidered", *Journal of Economic Theory*, 64, 1994, pp. 298-300.

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1. Introduction

Kranich (1988) establishes the interesting result that in a Walrasian model with voluntary transfers among altruistic agents, equilibrium is generally not Pareto efficient. He interprets this result to mean that: "… when agents wish to effect an equitable distribution of wealth, the Walrasian mechanism is ill suited for allocating resources". Kranich (1988; pg. 369). In a follow-up piece Kranich (1994) shows that in a one commodity, three agent gift-economy in which agents are *anonymous* and *self-biased,* equilibrium is generally not Pareto optimal and the First Welfare Theorem fails.

Stimulated by this result we use piecemeal reform methodology[1] to identify directions of Pareto improvements in the sort of economy studied by Kranich (1994). Section 2 presents the argument and main results. Section 3 presents some conclusions.

2. Model and results

Following Kranich (1994) we study a one good, three agent economy in which each agent has preferences defined over the entire commodity allocation. Allocations are denoted by (x_1, x_2, x_3) where x_i is the amount of good allocated to agent i. Agent i's preferences are represented by a continuous, differentiable, strictly quasi-concave, increasing utility function $u^i: \mathfrak{R}^3_+ \rightarrow \mathfrak{R}$. The following two assumptions are also made about preferences.

(A.1) *Self-biasedness*: $\forall x \in \mathfrak{R}^3_+$ and $\forall j \neq i$, $x_j = x_i \Rightarrow (\partial u^i/\partial x_i)(x) > (\partial u^i/\partial x_j)(x)$.

(A.2) *Anonymity*: $\forall x \in \mathfrak{R}^3_+$ and $\forall j \neq k \neq i$, $u^i(x) = u^i(\pi_{jk}(x))$ where $\pi_{jk}(x)$ is the allocation obtained by transposing the jth and kth coordinates of x.

[1] Piecemeal reform methodology is discussed in Diewert (1978), Dixit (1979), Guesnerie (1995), Myles (1995), and Blackorby (1999).

A *feasible gift* at allocation $x \in \Re^3_+$ is a vector $g^i = (g^i_1, g^i_2, g^i_3)$ such that $0 \leq x_i + g^i_i \leq x_i$, $g^i_j \geq 0$ for $j \neq i$ and $\Sigma_{j=1,3}\ g^i_j = 0$. Since utilities are differentiable and defined over entire allocations we have $u^1 = u^1(x_1, x_2, x_3)$, $u^2 = u^2(x_1, x_2, x_3)$, $u^3 = u^3(x_1, x_2, x_3)$ and:

$$
\begin{aligned}
du^1 &= (\partial u^1/\partial x_1)dx_1 + (\partial u^1/\partial x_2)dx_2 + (\partial u^1/\partial x_3)dx_3 \\
du^2 &= (\partial u^2/\partial x_1)dx_1 + (\partial u^2/\partial x_2)dx_2 + (\partial u^2/\partial x_3)dx_3 \\
du^3 &= (\partial u^3/\partial x_1)dx_1 + (\partial u^3/\partial x_2)dx_2 + (\partial u^1/\partial x_3)dx_3
\end{aligned} \tag{1}
$$

A redistribution (dx_1, dx_2, dx_3) with $dx_1 + dx_2 + dx_3 = 0$ is a *weak Pareto improvement, WPI,* if $\forall i = 1, 2, 3$, $du^i \geq 0$ and $du^i > 0$ for at least one i. A resistribution is a *strong Pareto improvement, SPI,* if $\forall i = 1, 2, 3$, $du^i > 0$. Clearly, the existence of a SPI entails the existence of a WPI.

From (A.2) we have $(\partial u^i/\partial x_j)(x) = (\partial u^i/\partial x_k)(x) \equiv u^i_{j\vee k}$, so (1) becomes:

$$
\begin{aligned}
du^1 &= (\partial u^1/\partial x_1)dx_1 + (u^1_{2\vee 3})(dx_2 + dx_3) \\
du^2 &= (\partial u^2/\partial x_2)dx_2 + (u^2_{1\vee 3})(dx_1 + dx_3) \\
du^3 &= (\partial u^3/\partial x_3)dx_3 + (u^3_{1\vee 2})(dx_1 + dx_2)
\end{aligned} \tag{2}
$$

The redistributions are of a single good so $dx_1 = -(dx_2 + dx_3)$ and (2) yields:

$$
\begin{aligned}
du^1 &= (\partial u^1/\partial x_1 - u^1_{2\vee 3})dx_1 \\
du^2 &= (\partial u^2/\partial x_2 - u^2_{1\vee 3})dx_2 \\
du^3 &= (\partial u^3/\partial x_3 - u^3_{1\vee 2})dx_3
\end{aligned} \tag{3}
$$

At any allocation there are nine possible relationships between the first partial derivatives of the individual utility functions:

$$\begin{aligned}
&(a)\ \partial u^1/\partial x_1 > u^1_{2\vee 3};\ (b)\ \partial u^2/\partial x_2 > u^2_{1\vee 3};\ (c)\ \partial u^3/\partial x_3 > u^3_{1\vee 2};\\
&(d)\ \partial u^1/\partial x_1 = u^1_{2\vee 3};\ (e)\ \partial u^2/\partial x_2 = u^2_{1\vee 3};\ (f)\ \partial u^3/\partial x_3 = u^3_{1\vee 2};\\
&(g)\ \partial u^1/\partial x_1 < u^1_{2\vee 3};\ (h)\ \partial u^2/\partial x_2 < u^2_{1\vee 3};\ (i)\ \partial u^3/\partial x_3 < u^3_{1\vee 2}.
\end{aligned} \tag{4}$$

We now identify combinations of these relationships between the marginal utilities of the agents which allow the existence of WPI's and SPI's. In the proof of our main proposition we also identify directions of redistribution that are welfare improving.

PROPOSITION: *If, in a one good three agent economy preferences are representable by continuous, differentiable, strictly quasi-concave, increasing utility functions over entire allocations which satisfy (A1) and (A2) then,*

(i) if either (abi) or (ahc) or (ahi) or (gbc) or (gbi) or (ghc) holds there exists a SPI;

(ii) if either (abf) or (aec) or (aef) or (aei) or (ahf) or (dbc) or (dbf) or (dbi) or (dec) or (dei) or (dhc) or (dhf) or (dhi) or (gbf) or (gec) or (gef) or (gei) or (ghf) holds there exists a WPI;

(iii) if either (abc) or (def) or (ghi) holds then no Pareto improving reallocation exists.

PROOF: *See Appendix.*

3. Conclusion

There are well known circumstances in which Walrasian equilibria are not Pareto optimal. These circumstances usually involve externalities, public goods or 'missing markets' of some sort. That the Walrasian mechanism can also be inefficient in a complete market context where agents are altruistic is surprising and interesting. It also raises the possibility that Pareto improving redistributions exist and can be characterised.

The task of this paper has been to identify relationships among individual preferences which permit the existence of Pareto improving reallocations in the single good model studied by Kranich (1994). A line for future research is to extend our investigation to the multi-good model of Kranich (1988).

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Appendix

PROOF OF THE PROPOSITION: Rewrite (3) in matrix notation where '*' will take values dictated by the expressions in (a) to (i) above.

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \qquad (5)$$

Case 1 (abc):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

There is no WPI or SPI because at least one $dx_i < 0$.

Case 2 (abf):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_3 < 0$ and $dx_1 \geq 0$, $dx_2 \geq 0$ with at least one strict inequality.

Case 3 (abi):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

SPI if $dx_3 < 0$ and $dx_1 > 0$, $dx_2 > 0$.

Case 4 (aec):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_2 < 0$, $dx_1 \geq 0$, $dx_3 \geq 0$.

Case 5 (aef):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_1 > 0$.

Case 6 (aei):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_1 > 0$ and $dx_3 < 0$.

Case 7 (ahc):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

SPI if $dx_i > 0$, $dx_2 < 0$, $dx_3 > 0$.

Case 8 (ahf):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_1 > 0$, $dx_2 < 0$.

Case 9 (ahi):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

SPI if $dx_1 > 0$, $dx_3 < 0$, $dx_3 < 0$.

Case 10 (dbc):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_2 > 0$, $dx_3 > 0$.

Case 11 (dbf):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_2 > 0$.

Case 12 (dbi):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_2 > 0$, $dx_3 < 0$.

Case 13 (dec):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_3 > 0$, $dx_1 \leq 0$, $dx_2 \leq 0$.

Case 14 (def):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

No WPI or SPI.

Case 15 (dei):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_3 < 0$.

Case 16 (dhc):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_2 < 0$, $dx_3 > 0$.

Case 17 (dhf):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_2 < 0$.

Case 18 (dhi):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_1 > 0$, $dx_2 < 0$, $dx_3 < 0$.

Case 19 (gbc):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

SPI if $dx_i < 0$, $dx_2 > 0$, $dx_3 > 0$.

Case 20 (gbf):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_1 < 0$, $dx_2 > 0$.

Case 21 (gbi):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

SPI if $dx_1 < 0$, $dx_2 > 0$, $dx_3 < 0$.

Case 22 (gec):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_i < 0$, $dx_3 > 0$.

Case 23 (gef):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_1 < 0$, $dx_2 + dx_3 = -dx_1$.

Case 24 (gei):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_1 < 0$, $dx_3 < 0$.

Case 25 (ghc):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & + \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

SPI if $dx_i < 0$, $dx_2 < 0$, $dx_3 > 0$.

Case 26 (ghf):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

WPI if $dx_1 < 0$, $dx_2 < 0$.

Case 27 (ghi):

$$\begin{bmatrix} du^1 \\ du^2 \\ du^3 \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

No WPI or SPI.

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