SAMUELSON'S 'CURIOUS CASE' REVISITED: IT TURNS OUT TO BE NORMAL – INDEED GENERIC

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Abstract

Samuelson (1974) noticed what he called a 'curious case' in which a redistribution of endowments, of the sort usually considered in connection with the Second Welfare Theorem, would not necessarily achieve a desired distribution of utilities. Samuelson's observation raises important questions for common interpretations of the Second Welfare Theorem. It also raises the interesting question of the sorts of redistributions that are needed to actually achieve a desired distribution of utilities. Motivated by Samuelson's work, this paper aims to do three things. Firstly, to explain why common interpretations of the Second Welfare Theorem are in jeopardy in the situation considered by Samuelson (1974). Secondly, to point out that far from being a 'curious case', the situation identified by Samuelson (1974) is normal (indeed generic). Thirdly to characterise an 'appropriate' redistribution of endowments to achieve a desired distribution of utilities when equilibrium is not unique.

Keywords: achieve, appropriate redistribution, support, uniqueness, welfare theorems.

JEL Classification: D51, D60.

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1. Introduction

Samuelson (1974) described what he called a 'curious case' in which a redistribution of endowments, of the sort usually considered in connection with the Second Welfare Theorem, would not necessarily achieve a desired distribution of utilities. Samuelson's observation raises important questions for common interpretations of the Second Welfare Theorem. It also raises the interesting question of the sorts of redistributions that are needed in order to achieve a desired distribution of utilities.

Motivated by Samuelson's work, this paper aims to do three things. Firstly, to explain why common interpretations of the Second Welfare Theorem are in jeopardy in the situation considered by Samuelson (11974). Secondly, to point out that far from being a 'curious case', the situation identified by Samuelson (1974) is normal (indeed generic). Thirdly to characterise an 'appropriate' redistribution of endowments to achieve a desired distribution of utilities when equilibrium is not unique. In order to achieve these objectives the paper is organized as follows. Section 2 introduces some basic definitions and summarises Samuelson's argument. Implications of the structure of the equilibrium set for common interpretations of the Second Welfare Theorem are then considered. We also characterise the sort of redistribution that is needed in order to achieve a desired distribution of utilities in economies where equilibrium is not unique. Our work here builds on the work of Samuelson (1974) and also of Bryant (1991, 1994) and Parrinello (1998). Section 3 presents some brief concluding remarks the primary one being that in an environment where equilibrium is not unique, non-market intervention will generally be needed in order to achieve a desired Pareto optimum.

2. Samuelson's 'curious case' and interpretations of the Second Welfare Theorem

2.1 Some basic concepts and definitions

A *private ownership production economy,* E, is made up of i = 1, …, I consumers, j = 1, …, J firms, ℓ = 1, …, L commodities. Each consumer is characterized by a non-empty *consumption set* $X_i \subset \Re^L$, a preference relation $\preceq_i$, a vector $\omega_i \in \Re^L$ of *initial endowments* and a *share of firm j,* θ_{ij} such that $0 \le \theta_{ij} \le 1$ and $\Sigma_i \theta_{ij} = 1$. Each firm j is characterized by a non-empty *production set* $Y_j \subset \Re^\ell$. The economy has a *total endowment* of commodities $\omega \in \Re^L$. An *exchange economy* is an E in which $Y_j = \emptyset$ for all j. At price vector p *aggregate excess demand* is $Z(p) = [\Sigma_i x_i(p, w_i) - \Sigma_j y_j(p) - \omega]$, where $x_i(p, w_i)$ is a preference maximizing vector of commodities for i, $y_j(p)$ is a profit maximizing vector for firm j and $w_i = p\omega_i + \Sigma_j\theta_{ij}py_j$ is the wealth of i. A *consumption allocation* (x_i^*) specifies for each i, a consumption vector $x_i^* \in X_i$. A *production allocation* (y_j^*) specifies for each j, a $y_j^* \in Y_j$. An allocation $[(x_i^*), (y_j^*)]$ is called *feasible* if $\sum_i x_i^* \le \sum_j y_j^* + \sum_i \omega_i$. A tuple $[(x_i^*), (y_j^*), p^*]$ and price vector is an *equilibrium* if for each i and j: (i) x_i^* is maximal relative to the preference order $\preceq_i$ in the set $B_i = \{x_i \in X_i : p^*x_i \le p^*\omega_i + \Sigma_j\theta_{ij}p^*y_j^*\}$; (ii) y_j^* maximizes profit so $p^*y_j \le p^*y_j^*$ for all $y_j \in Y_j$; (iii) $\sum_i x_i^* = \omega + \sum y_j^*$ so that all markets clear; (iv) $p^* \ne 0$. Let P denote the set of strictly positive normalised price, so $P = \{p \in \Re^L: p_1 > 0, \ldots, p_{L-1} > 0, p_L = 1\}$. The *equilibrium set* W for an economy E consists of the tuples $\{p, X_i, \preceq_i, Y_j, \theta_{ij}, \omega\}$ which satisfy the equation $\sum_i x_i(p, w_i) - \sum y_j(p) - \omega = 0$.

With these basic concepts in place, the essence of Samuelson's argument may be summarised as follows. Consider the exchange economy in Figure 1, which starts with an allocation of the social endowment at $\mathbf{w_1}$ (achieved through a redistribution from the initial allocation in the economy at $\mathbf{w_0}$).

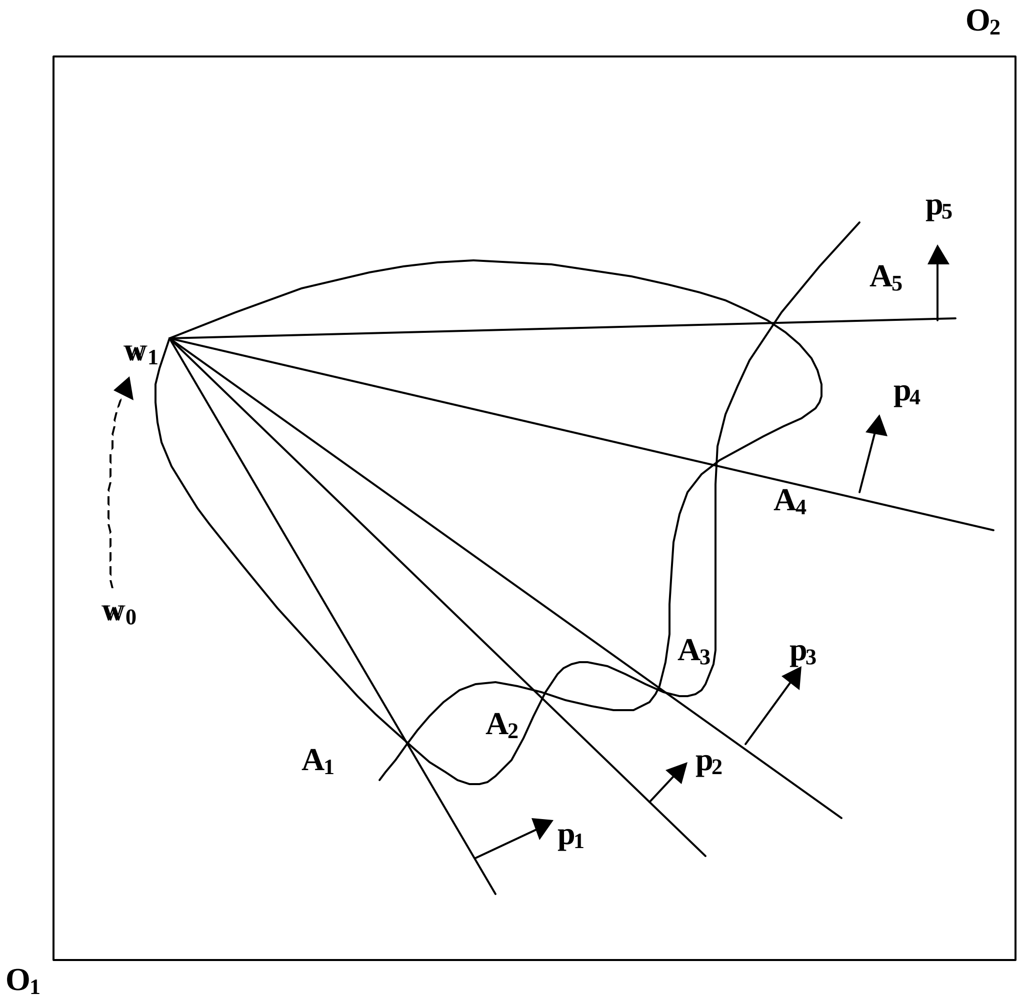

Figure 1: The Second Welfare Theorem and non-uniqueness

Competitive equilibrium occurs at price vectors $\mathbf{p^1}$, $\mathbf{p^2}$, $\mathbf{p^3}$, $\mathbf{p^4}$, $\mathbf{p^5}$. These prices support the Pareto allocations $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$, $\mathbf{A_4}$, $\mathbf{A_5}$. Suppose that allocation $\mathbf{A_3}$ is regarded as desirable from an equity point of view. Is it true that this Pareto optimum will necessarily be achieved? As Samuelson (1974) notes, the answer is clearly 'no' because there is nothing to prevent the price mechanism from selecting $\mathbf{p^1}$, $\mathbf{p^2}$, $\mathbf{p^4}$ or $\mathbf{p^5}$ and thereby decentralizing allocations $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_4}$ or $\mathbf{A_5}$. What is happening in this diagram may be further illustrated through an example due to Mas-Colell, Whinston and Green (1995; pp. 521 – 22). Consider a two person exchange economy with the following utilities and endowment vectors: $u_1(x_{11}, x_{12}) = x_{11} - 1/8x_{12}^{-8}$, $\omega_1 = (2, r)$ and $u_2(x_{21}, x_{22}) = -1/8x_{21}^{-8} + x_{22}$, $\omega_2 = (r, 2)$. Then the offer curves for these two individuals are:

$$OC_1(p_1, p_2) = [2 + r(p_2/p_1) - (p_2/p_1)^{8/9}, (p_2/p_1)^{-1/9}]$$
$$OC_2(p_1, p_2) = [(p_1/p_2)^{-1/9}, 2 + r(p_1/p_2) - (p_1/p_2)^{8/9}] \quad (1)$$

In the case where $r = 2^{8/9} - 2^{1/9}$, the three competitive equilibria are: $p_1/p_2 = 2/1, 1/1, ½$. Suppose that the allocation $[(1 + r, 1), (1, 1 + r)]$ is desired on distributional grounds. Then there is a Walrasian equilibrium and initial distribution that *supports* this allocation, namely $(p_1, p_2) = (1, 1)$ and $\omega = [(2, r), (r, 2)]$. However, as Figure 1 makes clear, there is nothing in the structure of the problem to ensure that the desired price equilibrium is selected. Hence there is no guarantee that 'letting the market work' will actually result in the desired allocation being achieved. Samuelson (1974) called this a 'curious case'. Notice also that 'perverse' demand behaviour is not needed to generate a situation like that in Figure 1. As Hara, Segal and Tadelis (1997; p. 15-7) show, even with normal offer curves[1] there can be multiple competitive equilibria. We now show that the case is in fact normal and indeed generic.

2.2 Generic non-uniqueness

Since the failure of the market to necessarily achieve the desired Pareto optimum in the case considered by Samuelson (1974) is a consequence of the non-uniqueness of competitive equilibrium, it is reasonable to ask whether non-uniqueness may be regarded as an uninteresting pathology, which may be safely ignored, or is something which must be allowed for as a phenomenon likely to occur in most market economies?

The uniqueness question for competitive equilibrium has been studied in a literature stretching from Wald (1936) to Jerison (1999). This research effort has established that uniqueness is generally not the case. As Debreu (1970) puts it: "...The uniqueness property has been obtained only under strong assumptions and economies with multiple equilibria must be allowed for." Debreu (1970; p. 387). Similar remarks are found in Dierker (1983), Kehoe (1985), Mas-Colell, Whinston and Green (1995) and the recent survey by Kehoe (1998).

Failing uniqueness, it is hoped that the equilibrium set is finite (and preferably small). Debreu (1970) established that almost all exchange economies do indeed have a finite number of locally isolated equilibria, provided preferences are smooth enough to yield continuously differentiable demands. Debreu's result is extended by Smale (1976), Fuchs (1977) to cover production economies and is obtained by Pascoa and da Costa

[1]Following Mas-Colell, Whinston and Green (1995, p. 541) and offer curve is called *normal* if an increase in the price of one commodity in a two commodity economy, leads to an increase in the demand for that commodity only if the demands for the two commodities both increase.

Werlang (1999) in the exchange case under the weakened assumption that preferences are representable by a continuous, monotone and strictly concave utility function whose bordered Hessian is non-singular almost everywhere.

Given that equilibrium is almost never unique, we now show that when equilibrium is not unique then, contrary to widespread interpretations of the Second Welfare Theorem, considerable non-market intervention will generally be needed in order to achieve any desired Pareto optimal allocation.

2.3 Interpreting the Second Welfare Theorem

Luenberger (1994) has observed that the Second Welfare Theorem is one of the three most important results in modern economics. In particular he writes: “The most remarkable achievements of modern microeconomic theory are the proof of the existence of an equilibrium and the First and Second Theorems of Welfare Economics … Understandably, therefore, there has been much attention devoted to various interpretations, alternative proofs, and extensions of these basic results.” Luenberger (1994; p. 147).

The status accorded the Second Welfare Theorem follows from the well known fact that while equilibrium allocations may be Pareto efficient (if, say, the conditions of the First Fundamental Theorem of Welfare Economics hold), it may nevertheless be the case that the operation of markets results in an ethically objectionable distribution of welfare among individuals in the economy. As Feldman (1987) notes, this possibility has lead some to advocate the total abandonment of the market mechanism and its replacement by central planning. Others have argued for various modifications of market prices in order to achieve distributional objectives through, for example, taxes and subsidies, price controls and similar interventions. Advocates of competitive markets generally respond to such suggestions by arguing for the retention of markets and by asserting that distributional concerns can be addressed without dispensing with, or significantly over-riding, the market mechanism. The result in economic theory which is appealed to in order to sustain such a point of view is the Second Welfare Theorem.

Typical of such arguments is the following one due to Mas-Colell, Whinston and Green (1995): "[The second welfare theorem] says that under convexity assumptions (not required for the first welfare theorem), a planner can achieve any desired Pareto optimal allocation by appropriately redistributing wealth in a lump-sum fashion and then 'letting the market work'. *Thus, the second welfare theorem provides a theoretical affirmation for the use of competitive markets in pursuing distributional objectives*." Mas-Colell, Whinston and Green (1995; p.524; emphasis added).

The belief annunciated by Mas-Colell, Whinston and Green (1995) that the Second Welfare Theorem provides theoretical justification for the use of markets to achieve distributional objectives is a belief that is widely held. For instance Varian (1987) similarly claims that: "...The Second Theorem of Welfare Economics asserts that under certain conditions, every Pareto efficient allocation can be *achieved* as a competitive equilibrium. What is the meaning of this result? The Second Welfare theorem implies that problems of distribution and efficiency can be separated ... whatever your criteria for a good or just distribution of Welfare, you can use competitive markets to *achieve* it..." Varian (1987; p. 502; emphasis added). In a complementary fashion Hammond (1990) asserts that: "... A much more promising defense of perfectly competitive markets is based, of course, on the second efficiency theorem of welfare economics [which states] any Pareto efficient allocation of resources in which no individual is on the margin of being forced below subsistence can be *reached* through perfectly competitive markets, provided that the invisible hand is supplemented by a suitable method for redistributing wealth." Hammond (1990; p. 8; emphasis added). Anderson (1988) summarizes the common elements in these remarks as he writes: "...consider the interpretation [usually] placed on the second welfare theorem ... It is asserted that it would be better for government to redistribute income, and then allow the workings of the market to determine the allocation of commodities to individuals rather than have the government establish subsidies for certain commodities or to allocate goods through non-market mechanisms." Anderson (1988; p. 361). Similar remarks to those above may be found in Gravelle and Rees (1981)[2], Shone (1975)[3], Russell and Wilkinson (1979)[4], Allingham (1983)[5], Cornwall (1984)[6], Feldman (1987)[7], Blad and Keiding (1990)[8], Kreps (1990)[9], Silberberg (1990)[10], Eaton and Eaton (1991)[11], Varian (1992)[12] and Starr (1997)[13].

[2] "The market economy can be thought of as an efficient 'black box' or resource allocation machine: feed in an initial wealth distribution, the mechanism churns away and out comes a Pareto Optimum ... A policy maker who dislikes the welfare distribution implied by a given market mechanism can best improve things *not* by interfering with the market mechanism - the works of the black box - but rather by changing the wealth distribution directly [since] the policy of lump sum redistribution together with unfettered operations of the market mechanism is superior [to any distortion of market prices]. [Indeed] it is possible to prove the following proposition: *every Pareto Optimal resource allocation can be achieved as a competitive market equilibrium given an appropriate initial distribution of wealth* (for a proof see Takayama (1974. p. 185-201))." Gravelle and Rees (1981; p. 485).

[3] "...The requirements of Theorem 10.5 [the Second Welfare Theorem] ensures the conditions under which a Pareto Optimal state *can always be* reached *by perfect competition*." Shone (1975; p. 262)

[4] "...The significance of the second fundamental optimality principle is that, given our assumptions, any Pareto-optimal allocation can be *achieved* by the decentralised decision making of consumers pursuing their own self interest in competitive markets. *Centralised planning is unnecessary*." Russell and Wilkinson (1979; p. 356)

[5] "...the second theorem of welfare economics [states that] any optimal allocation may be *obtained* as an equilibrium allocation, given the appropriate endowment allocation." Allingham (1983; p. 28)

We now examine these claims in the light of Samuelson's 'curious case' and advance the argument that there are general circumstances in which they are not well founded. We do this by showing that the Second Welfare Theorem does not admit the interpretation placed on it in an environment where equilibrium is not unique. Since this turns out to be most economic environments, it follows that the Second Welfare Theorem does not constitute a foundation in economic theory for the particular approach to achieving distributional objectives noted above, and for which the Second Welfare Theorem is used as justification. We also show that when equilibrium is not unique the 'appropriate' redistribution of endowment (or wealth) alluded to in the quotations above

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[6] "...the result of this section gives a much deeper result than the previous section. It can be paraphrased: if certain assumptions including convexity hold, then all Pareto efficient outcomes can be *gotten* as outcomes of a market process very similar to competitive equilibrium." Cornwall (1984; p. 382-3)

[7] "...The Second Fundamental Theorem of Welfare Economics established that the market mechanism, modified by the addition of lump sum transfers, can *achieve* virtually any desired optimal distribution." Feldman (1987; p. 891)

[8] "It turns out that Theorem 4.3 has a converse: all Pareto optimal allocations can be *obtained* through the market." Blad and Keiding (1990; p. 124)

[9] "... In other words if one imagines that it would be difficult for a social dictator to find an equitable and efficient allocation of the social endowment, and if the economy has well functioning markets, the dictator might choose to reallocate initial endowments in an equitable fashion and then *let the market take over*." Kreps (1990; p. 200)

[10] "...The [Second Welfare Theorem] is the statement that there is an allocation under perfect competition for any overall Pareto optimum. That is, starting now with a point on the Pareto frontier, there exists a competitive solution which *achieves* that optimum." Silberberg (1990; p. 588)

[11] "Suppose that we have identified some Pareto-optimal allocation that we would like to implement. The second theorem tells us first to redistribute the initial endowment and then to rely on competitive markets to *achieve* Pareto optimality." Eaton and Eaton (1991; p. 421)

[12] "...the above proposition [Second Welfare Theorem] shows that every Pareto efficient allocation can be *achieved* by a suitable reallocation of wealth." Varian (1992; p. 346)

[13] "The Second Fundamental Theorem of Welfare Economics says that … [a]ny desired redistribution of welfare (subject to attainability) can be achieved through a market mechanism, subject to a redistribution of endowment and ownership.' Starr (1997; p. 146).

generally leaves little or no room at all for market prices in the task of achieving a desired Pareto allocation.

2.4 What the SFTWE actually says: 'supporting' Pareto allocations

As noted above, it is widely believed: (i) that the market mechanism can be used to *achieve* any desired distribution of welfare, and (ii) that the Second Welfare Theorem justifies such a belief. We now show that there are general circumstances in which these interpretations of the Second Welfare Theorem are not correct. We begin by considering exactly what it is that the Second Welfare Theorem asserts.

Theorem [Debreu (1959)]: *Let* $E = \{ X_i, \preceq_i, w_i, Y_j, q_{ij}, \ell \}_{i=1}^{n}{}_{j=1}^{m}$ *be a complete market Arrow-Debreu economy and let* (x^*, y^*) *be a Pareto optimal allocation at which at least one consumer is not satiated. If E is such that:*
(i) for all i, X_i *is convex;*
(ii) for all i, the preference relations $\preceq_i$ *are convex and continuous;*
(iii) the set $\sum_j Y_j$ *is convex,*
then there is a price system $p^* \neq 0$ *such that:*
(a) x_i^* *minimizes* $p^* x_i$ *on* $\{x_i \in X_i : x_i \succeq_i x_i^*\}$ *for every i;*
(b) y_j^* *maximizes* $p^* y_j$ *on* Y_j *for every j.*
Thus under conditions (i)-(iii) there exists a competitive equilibrium price system which supports the Pareto allocation (x^*, y^*).

Proof : See Debreu (1959; pg. 96). For a sketch proof see Figure 2.

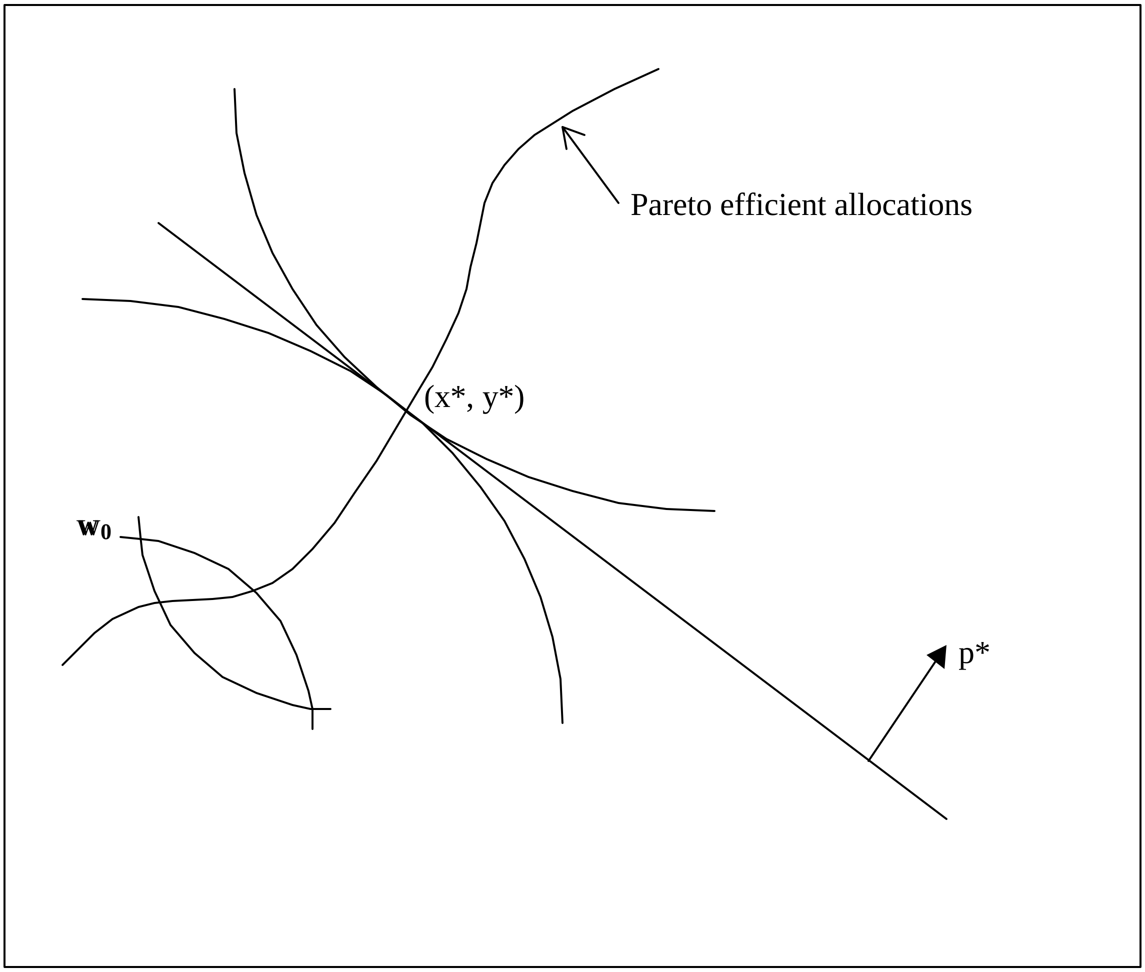

Figure 2: Sketch proof of the Second Welfare Theorem

Remark. The classical version of the Second Welfare Theorem, stated above, has been extended in various directions. Winter (1969) shows that in the absence of gifts the theorem holds in an economy with a benevolent non-participatory agent. Archibald and Donaldson (1976) extend the theorem, replacing Winter's non-malevolent preferences assumption with a non-participation assumption (along with a strong preference separability condition). Rader (1980) generalizes the theorem to the case where preferences are interdependent. John and Ryder (1985) establish the theorem an exchange economy where preferences are locally satiated and in a production economy provided at least one consumer has non-satiated preferences. They also show that the conclusion of the theorem fails if all preferences are satiated. Kahn and Vohra (1987) generalize to arbitrary non-convex production environments the partial generalizations achieved by Foley (1970) and Guesnerie (1975). Mas-Colell (1987) establishes a version of the theorem in the context of a large economy in which where net trades are anonymous. Kranich (1988) shows that in the presence of ‘general’ altruism, the theorem fails, a result similar to that established by Goldman (1978) for a gift economy. Kranich (1994) shows that in Goldman’s model if ‘symmetry’ holds, so that the identity of agents is not known, and there is a slight bias on the part of individuals towards their own consumption, then the theorem is restored in Goldman’s model. Anderson (1988) generalizes the theorem to cover non-convex preferences. In a temporary equilibrium framework Allard, Bronsard and Richelle (1989) show that the theorem holds if expectations are strongly Roy consistent, but fails if expectations are only weakly Roy consistent. Pan (1995) demonstrates that a constrained Pareto optimum can be supported by competitive prices in an incomplete markets context, after a suitable redistribution of first period endowments. Diamantaras, Gilles and Scotchmer (1996) extend the theorem to economies with public projects, nonessential private goods and production under conditions of convex costs. Malcolm (1998) shows that the theorem generalizes to environments where production technologies are non-convex and preferences are non-convex as well.

It is important to note that the Second Welfare Theorem, in either its classical or extended form, is an *existence theorem* for a particular sort of price system. What the theorem guarantees is the existence of a competitive equilibrium which *supports* any given Pareto optimal allocation, provided the economy satisfies certain conditions. Also the theorem *does not* claim that the supporting equilibrium price vector is necessarily an equilibrium for a given economy, in particular, for the current distribution of endowments. For instance, if the initial allocation is ω_0 in Figure 2 and the desired Pareto optimum is at the allocation (x*, y*) then the current economy does not have a competitive equilibrium which supports (x*, y*). This fact is well known, but in conjunction with non-uniqueness of equilibrium it has important implications for what constitutes an 'appropriate' initial redistribution of endowment (or wealth) and the role of markets in achieving any desired Pareto allocation, an implication we now discuss[14].

[14] We note in passing a further problem with common interpretations of Second Welfare Theorem, even if equilibrium were unique, and this occurs in the event where the price adjustment process is not globally stable. When global stability fails, it again cannot be claimed that a market mechanism will *achieve* a desired Pareto optimum, precisely because the market process may fail to arrive at any equilibrium price vector at all. It is therefore reasonable to ask what is known about the stability of market mechanisms. Research stretching from Walras (1874) to Herrmann and Kahn (1999) via Saari and Simon (1978), Hahn (1982), Jordan (1982), Saari (1985) Ingrao and Israel (1990) and Saari (1995), Bryant (1996), Keisler (1996), Rader (1996a) indicates that the stability of equilibria, relative to an informationally reasonable adjustment process, may be difficult to guarantee in general. Particularly interesting in this connection is the work by Saari-Simon (1978) and Saari (1985) which characterise necessary conditions for effective (continuous-time or iterative) price mechanisms. Recalling that the only processes which are guaranteed to be globally stable are those of the Global Newton type we note an important feature of such processes, namely, that while such processes are successful in selecting *a point* in the equilibrium set there is generally no relationship between the starting point and the end point for such processes. As Herrmann and Kahn (1999) note: "… the mechanism computes *an* equilibrium (among a finite/discrete set of equilibria) that has *no* relation to the starting point of the process. The later drawback means that GNM can say nothing about which equilibria is computed and why it was: that is there is no control over which equilibrium is selected." Herrmann and Kahn (1999; p. 422-23).

2.5 Appropriate redistribution when equilibrium is not unique

In light of the observation in the previous paragraph, the Second Welfare Theorem is sometimes written in such a way as to make explicit the likely need for some sort of non-market intervention, often in the form of an initial redistribution of endowment or wealth. A typical reformulation of the theorem along these lines is:

Theorem [Mas-Colell, Whinston and Green (1995)]: *If household preferences and firm production sets are convex, all preferences are locally non-satiated, there is a complete set of markets with publicly known prices, and if every agent acts as a price taker, then any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth or endowments are arranged.*

Proof : See Mas-Colell, Whinston and Green (1995; pg. 308 and pp. 552-554). For a sketch proof, also provided by Mas-Colell, Whinston and Green (1995; p. 525), see Figure 3.

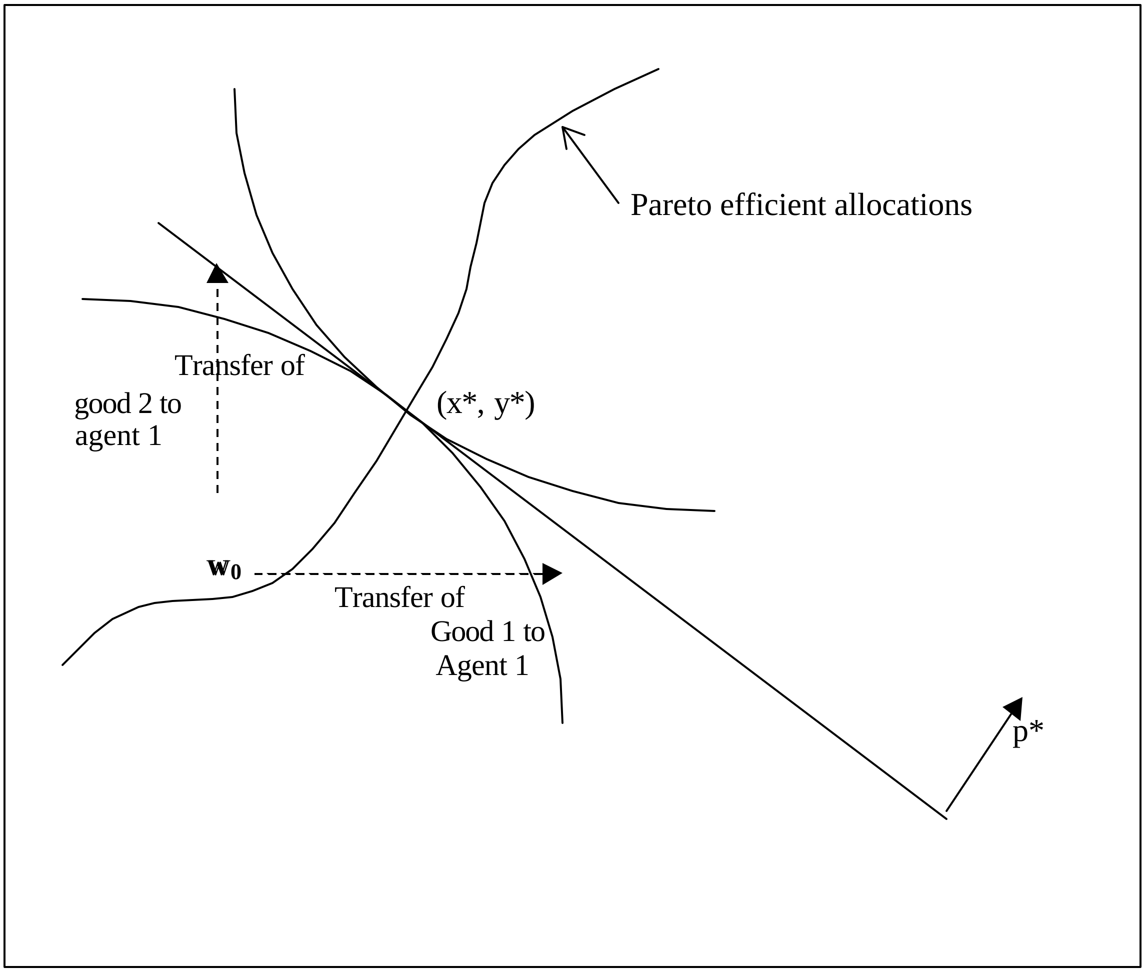

Figure 3: A Pareto optimum supported by endowment redistribution and prices

Remark. According to Mas-Colell, Whinston and Green (1995), the redistribution needs only finish at some point in $\{p\}^{\perp}$. This is the hyperplane defined by p which supports the allocation (x*, y*). It is claimed by these authors that provided the redistribution lodges anywhere in $\{p\}^{\perp}$ it will be 'appropriate' in that it will allow the actual decentralization of the desired Pareto optimum. This is not generally the case. We now show that in an economy where equilibrium is not unique this is not the case and that a much stricter requirement for an 'appropriate' redistribution has to be satisfied.

2.5 'Appropriate' redistributions that guarantee 'achieved'

It is known from Debreu (1970) and others that there are finitely many locally isolated equilibria for most exchange economies. This means that the typical economy has an offer curve map which looks qualitatively like that in Figure 1. For fixed preferences an economy is defined by the location of the initial endowment, ω and since associated with any allocation of the initial endowment, apart from a Pareto allocation, there is usually more than one equilibrium, it follows that in order to *achieve* any desired Pareto optimum the *appropriate* redistribution of initial endowments involves actually taking the economy directly to the desired Pareto optimum or at least into a neighbourhood of the desired allocation. If this is not done then, as Figure 1 shows, there is a possibility that the market will select the 'wrong' final equilibrium distribution od utilities. We formalise these remarks in the following results.

Theorem 1: *Consider a complete market economy* $\mathcal{E}$ *in which every agent acts as a price taker,* x^* *is a Pareto optimal allocation with* $\sum_{i \in I} x_i^* \gg 0$ *and* ω *is the initial endowment distribution. If:*

- *(i)* $X_i \subseteq \mathbb{R}^\ell$ *is non-empty and convex;*
- *(ii) for all i,* $\preceq_i$ *is defined on* X_i *and is complete, continuous, reflexive transitive strictly convex, strongly monotonic and representable by a utility function,* u_i*;*
- *(iii) for all i, the topological boundary* $\partial\!\preceq_i$ *of* $\preceq_i$ *is a* C^2 *manifold so that the utility function* u_i *is* C^2 *with a nowhere vanishing derivative;*
- *(iv) the aggregate production set, Y, is convex;*
- *(v) the set of feasible allocations is compact;*

then (1) there is a $p^* \in \mathbb{R}^\ell$ *such that* $p^* \neq 0$*, and* p^* *is a Walrasian equilibrium price vector which supports* x^**. If the initial endowment for* $\mathcal{E}$ *is such that* $\omega \in \{x \in \mathbb{R}^\ell : p^*x = p^*x^*\}$ *then (2) there are finitely many locally isolated Walrasian equilibria associated with* ω *and if* $\omega \in \{[x \in \mathbb{R}^\ell : p^*x = p^*x^*] \setminus x^*\}$ *then the number of such equilibria is almost always greater than one. If* $\omega = x^*$ *then (3) there is a unique Walrasian equilibrium which supports* x^**.*

Proof : (See Appendix).

A second result can be established which shows that the potential distributional error from using markets can't increase as the redistribution approaches the desired Pareto allocation along the hyperplane which supports the desired allocation. In the limit the error goes to zero as established in Theorem 1. In order to facilitate the proof we make the following definition.

Definition 9.1 (Trading lens): The *trading lens* for the endowment ω_k, $1 \leq k \leq n$, is the intersection of the upper-contour sets for each consumer in the economy relative to ω_k i.e. $TL(\omega_k) = \cap_{i\in I} \{x_i \in X_i : \omega_k \preceq_i x_i\}$.

Theorem 2: *Consider a complete market economy* E *in which every agent acts as a price taker,* ω^* *is a Pareto optimal allocation with* $\sum_{i\in I} \omega_i^* >> 0$ *and* ω_1 *is the initial endowment distribution. Let* $\omega_n \in \{\omega : p\omega_n = p\omega_1\}$ *be a sequence of redistributions along the hyperplane which supports the desired Pareto optimum* ω^*. *If* ω_n *is a sequence such that* $TL(\omega_n) \subset TL(\omega_{n-1})$ *for all* $n \geq 2$ *and if:*

- *(i)* $X_i \subset \mathbb{R}^\ell$ *is non-empty and convex;*
- *(ii) for all i,* $\preceq_i$ *is defined on* X_i *and is complete, continuous, reflexive transitive strictly convex, strongly monotonic and representable by a utility function,* u_i;
- *(iii) the aggregate production set, Y, is convex;*
- *(iv) the set of feasible allocations is compact;*
- *(v) the set of equilibria associated with the endowment* ω_n *is* $E(\omega_n)$,

then $E(\omega_n) \subseteq E(\omega_{n-1})$ *for* $n \geq 2$ *and* $E(\omega^*)$ *has one element,* $\{p,\omega^*\}$, *the no-trade equilibrium allocation.*

Proof : (See Appendix).

Remark 1. We may illustrate the conclusion of this theorem as follows. Suppose the desired Pareto optimal allocation is $\mathbf{A_3}$ in Figure 4. Then if the initial redistribution which is adopted takes the economy from $\mathbf{w_0}$ to $\mathbf{w_1}$ then the operation of markets may result in the desired allocation being missed by a 'large distance' (e.g. allocations at $\mathbf{A_1}$ or $\mathbf{A_5}$ might result, instead of the desired allocation $\mathbf{A_3}$**)**. However, by construction the offer curve of an individual must lie in the upper contour set defined by the indifference curve for that person through the initial endowment point.

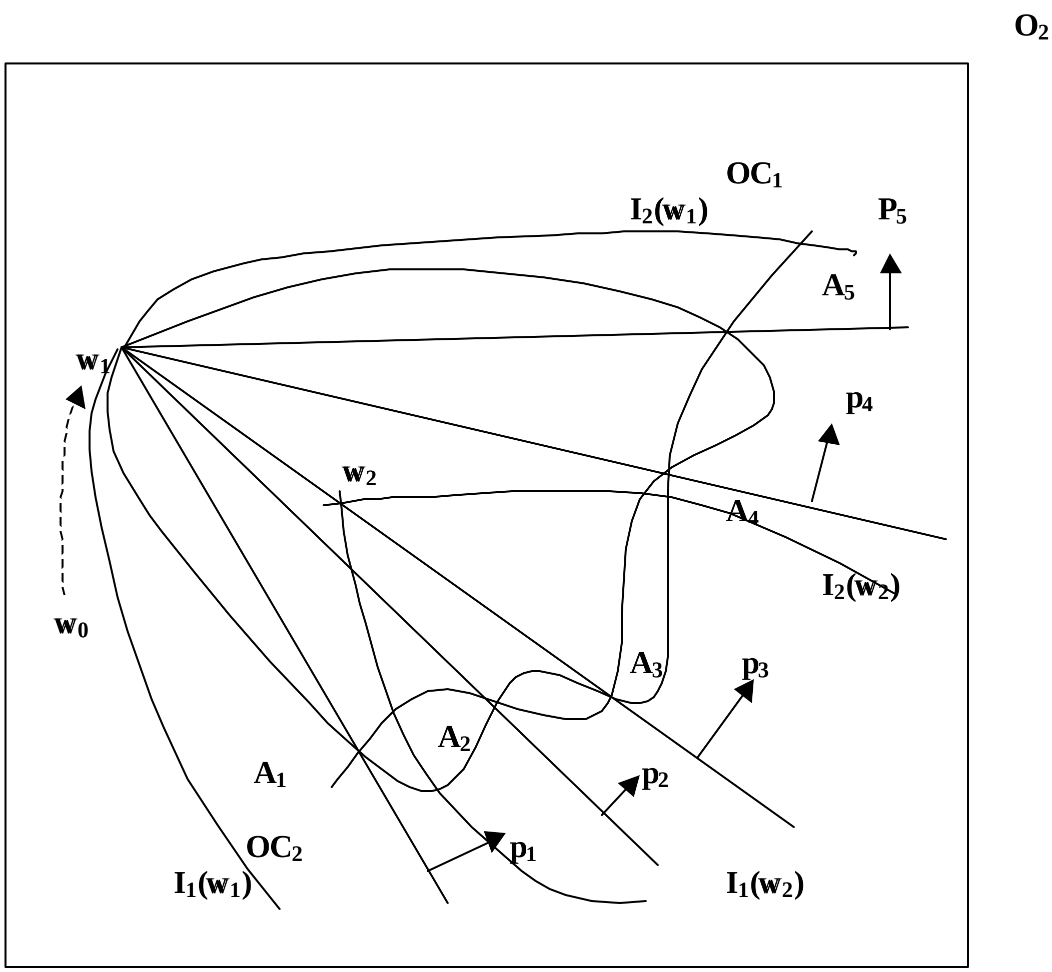

Figure 4: Approximate achievement of a desired allocation

It follows that as the redistribution gets closer to the final desired allocation the set of possible equilibria can't increase and the size of the possible error in achieving the desired allocation can't increase either, provided the allocation stays in the lens formed by the two individual indifference curves through the endowment point (and certainly if it stays on the hyperplane supporting the desired allocation). This can be seen in Figure 4. The equilibria associated with the distribution of endowments ω_1 will support allocations $\mathbf{A_1}$**,** $\mathbf{A_2}$ $\mathbf{A_3}$ $\mathbf{A_4}$ or $\mathbf{A_5}$. However, if the starting point is the allocation ω_2 then the equilibria must be contained in the lens formed by the indifference curves $I_1(\omega_2)$ and $I_2(\omega_2)$. Thus the equilibria, wherever they are exactly (and they have not been drawn in here because the diagram is already busy enough), will be closer to $\mathbf{A_3}$ than either $\mathbf{A_1}$ or $\mathbf{A_a}$ or $\mathbf{A_5}$. In this diagram, the closer the initial allocation comes to $\mathbf{A_3}$ the better the approximation, until it is exact, when it is at the desired Pareto optimum, as the theorem asserts.

Remark 2. If enough conditions are put on the economy so that the equilibrium that supports the desired Pareto optimum is locally stable then in an economy with a multiplicity of equilibria (i.e. the general case), then 'achieved' can be validly substituted for 'supported' in the SFTWE if the appropriate redistribution takes the economy into the neighbourhood of the desired Pareto optimal allocation. However this leaves little room for 'letting the market work' in the pursuit of distributional objectives, since all the work in achieving the desired distribution of welfare has been done by the non-market act of redistribution. Consequently, in all these cases the argument of Mas-Colell, Whinston and Green (1995), and others listed above, that the SFTWE guarantees that a planner can *achieve* any desired Pareto optimal allocation by appropriately redistributing wealth in a lump-sum fashion and then ‘letting the market work’ may be an overstatement. The SFTWE provides a theoretical affirmation for the use of competitive markets in pursuing distributional objectives in economies with multiple equilibria, only when the ‘appropriate’ redistribution involves taking the economy directly to (or at least into the neighbourhood of) the desired Pareto optimum.

3. Conclusion

The uniqueness question for competitive equilibrium has been studied in a literature stretching from Wald (1936) to Jerison (1999). The major conclusion reached by that research effort is that economies with multiple equilibria are the norm and that uniqueness holds only under restrictive and implausible conditions.

In the absence of uniqueness, it is of interest to know something about the cardinality and structure of the equilibrium set. Debreu (1970) established that for almost every exchange economy the equilibrium set is finite and the equilibria are locally isolated. Smale (1976), Fuchs (1977) and Pascoa and da Costa Werlang (1999), extend Debreu’s original result in various directions.

The non-uniqueness of equilibrium has important implications for common interpretations of the Second Welfare Theorem. In particular it is often claimed that any desired Pareto optimum can be achieved by markets, possibly preceded by some initial redistribution of endowments (or wealth) and that the Second Welfare Theorem provides theoretical justification for such a claim. Since the hypotheses of the Second Welfare Theorem do not guarantee uniqueness of competitive equilibrium, the Second Welfare Theorem does not provide the claimed justification for the use of markets in the pursuit of distributional objectives. We conclude by giving a set of conditions under which the cited interpretation of the Second Welfare Theorem are true. An interesting thing about our result is that in an economy where equilibrium is not unique, all the work in achieving a desired Pareto optimum is done by non-market means. In particular it is necessary to make an initial redistribution directly to the desired Pareto allocation, or at least to a neighbourhood of the desired Pareto allocation thereby leaving little or nothing for the market mechanism to do. Only in the case where equilibrium is unique (and globally stable) are standard interpretations of the Second Welfare Theorem and standard characterisations of appropriate redistributions generally valid.

Appendix

Proof of Theorem 1.

(1: Existence of Pareto supporting price equilibria)

For each i, let $S_i = \{x_i \in X_i : x_i^* \preceq_i x_i\}$. Since by (ii) preferences are continuous and convex, each S_i is closed and convex. Also each $S_i \neq \emptyset$ since $x_i^* \in S_i$. Let $S \equiv \sum_i S_i$. Then $S = \{z : z = \sum_i x_i, x_i \in S_i\}$ and since $S_i \subset X_i \subset \Re^\ell$, S is closed, convex and nonempty and consists of all aggregate bundles that have allocations that are preferred by all individuals to the allocation x*. Let $y^* = \sum_i x_i^*$. By definition $y^* \in S$ and by Pareto efficiency, y* is a boundary point of S, i.e. $y^* \in bdS$. Suppose not i.e. suppose $\exists y' < y^*$ with $y' \in S$. Then there would also be $x' = (x_1', x_2', \ldots x_n')$ with $\sum_i x_i' = y'$ and $x_i^* \preceq_i x_i'$ for all i. Then the allocation defined by $x_i'' = x_i' + (y^* - y')/n$ would, by strong monotonicity (ii), satisfy the condition: $x_i^* \preceq_i x_i' \prec_i x_i''$ and $\sum_i x_i'' = y^*$. But this contradicts the Pareto efficiency of x*. Since $y < y^*$ implies $y \notin S$ it follows that $y^* \in bdS$. Minkowski's separating hyperplane theorem guarantees that if C is a convex set and $k \in bdC$ then there is a hyperplane containing k and enclosing C in one of its closed half spaces. Therefore there is a hyperplane through y* with S contained in one half space and Y contained in the other. This means $\exists p^* \in \Re^\ell$ such that $p^*z \geq p^*y^*$ for all $z \in S$ and $p^*y \leq p^*y^*$ for all $y \in Y$. Let ι_k be the k^{th} unit vector in $\Re^\ell$ and let ι be an ℓ vector of 1's. By strict monotonicity (ii), the aggregate bundle $y^* + \iota_k \in S$, since the additional amount of the k^{th} commodity can be distributed among the n consumers making them all better off. Thus $p^*(y^* + \iota_k) \geq p^*y^* \Leftrightarrow p^*\iota_k \geq 0$ for any k. Thus $p^* \geq 0$. By continuity of preferences (ii), and $y^* > 0$, this argument can be extended to infer that $y^* + \iota_k - \varepsilon\iota \in S$ for small $\varepsilon > 0$. Therefore we have

$p^*(y^* + \iota_k - \varepsilon\iota) \geq p^*y^*$ or equivalently $p^*\iota_k \geq \varepsilon p^*\iota > 0$. Thus $p^* > 0$ and $p^* \neq 0$ is established. Now suppose $x_i^* \prec_i x_i$ for some i. Then by strict monotonicity $x_i \neq 0$. Let $x' = (x_1', x_2', \ldots x_n')$ where $x_i' = (1 - \varepsilon)x_i$ and $x_j' = x_j^* + (\varepsilon / n - 1)x_i$ for $j \neq i$, obtained by distributing part of x_i among the other individuals. By continuity and strong monotonicity of each $\preceq_i$, $\exists\varepsilon > 0$ such that this allocation is preferred to x* by all individuals. Then $\sum_j x_j = x_i + \sum_{j \neq i} x_j^* = x_i + (y^* - x_i^*)$ and this is in S. Thus we have $p^*(x_i + y^* - x_i^*) \geq p^*y^* \Leftrightarrow p^*x_i \geq p^*x_i^*$. However, given that $x_i^* \prec_i x_i$ then $\exists\alpha \in (0, 1)$ such that $x_i^* \prec_i \alpha x_i$ and $\alpha p^*x_i \geq p^*x_i^*$. But because $p^*x_i > 0$ it follows that $p^*x_i > p^*x_i^*$. Therefore, $u(x_i) > u_i(x_i^*) \Rightarrow p^*x_i > p^*x_i^*$ which is the condition for consumer equilibrium in a Walrasian state. Since $p^*y \leq p^*y^*$ for all $y \in Y$ because of the original definition of p* as part of a hyperplane separating S and Y, the profit maximisation condition for producers in a Walrasian state is satisfied. Thus we have established the existence of a $p^* \in \Re^\ell$ such that $p^* \neq 0$, and p* is a Walrasian equilibrium price vector which supports x*.

(2: Finiteness and local uniqueness of equilibria)

Under conditions (iii), (iv) Smale (1974) shows that there is an open dense set of economies Θ, in the space of all economies, for which each element has locally unique equilibria. Further, if $E \in \Theta$ and (v) holds, then the equilibrium set is finite. From our work in Chapter 7 we know that under the conditions assumed in the theorem, the equilibrium set generally has more than one element.

(3: Uniqueness of equilibrium at a Pareto optimum)

Suppose the initial endowment allocation $\omega = (\omega_1, \omega_2, \ldots \omega_n)$, i.e. the no-trade point, is a Walrasian equilibrium. We will show that this is the unique equilibrium allocation. To fix ideas we start with the exchange economy case and then extend the argument to the production case. Let an allocation $x = (x_1, x_2, \dots x_n)$ and a price vector p constitute a Walrasian equilibrium when the distribution of endowments is $\omega = (\omega_1, \omega_2, \dots \omega_n)$. Since ω_i is affordable at p for each consumer i, we have $\omega_i \preceq_i x_i$ for all i. But from the FFTWE, $(\omega_1, \omega_2, \dots \omega_n)$ is a Pareto optimum and so it must be that $\omega_i \sim_i x_i$ for all i. But then it must be that $x_i = \omega_i$ for all i because otherwise by (ii), particularly the strict convexity of preferences, the allocation $(1/2x_1 + 1/2\omega_1, 1/2x_2 + 1/2\omega_2 \dots. 1/2x_n + 1/2\omega_n)$ would be Pareto superior to $(\omega_1, \omega_2, \dots \omega_n)$. For production let $(p, x_1, x_2, \dots x_n, y_1, y_2, \dots y_m)$ be an equilibrium price vector and allocation such that $x_i \in Y + \{\omega\}$. Let $(p', x'_1, x'_2, \dots x'_n, y'_1, y'_2, \dots y'_m)$ also be an equilibrium with $x'_i \in Y + \{\omega\}$. We show that $x_i = x'_i$ for every i and that $p = p'$. Since $(p', x'_1, x'_2, \dots x'_n, y'_1, y'_2, \dots y'_m)$ is assumed to be a Walrasian equilibrium it must be that $p'x_i \leq p'\omega_i + \sum_j \theta_{ij}p'y'_j$. Thus by the utility maximisation condition for equilibrium $x_i \preceq_i x'_i$ for every i. Since by (v) Y is convex , the consumption allocation $(1/2x_1 + 1/2x'_1, 1/2x_2 + 1/2x'_2 \dots. 1/2x_n + 1/2x'_n)$ is feasible. Further, by strict convexity of $\preceq_i$ if $x_i \neq x'_i$ then $x_i \prec_i 1/2x_i + 1/2x'_i$. However, because preferences are strongly monotone the FFTWE holds so if $(p, x_1, x_2, \dots x_n, y_1, y_2, \dots y_m)$ is a Walrasian equilibrium then there is no i for which a feasible Pareto superior allocation exists. Therefore there is no i for which $x_i \prec_i 1/2x_i + 1/2x'_i$. Thus $x_i = x'_i$ for each i. Also by strict convexity of preferences and the fact that $x_i = x'_i$ for each i we have $p = p'$.

Remark. Part 1 of the proof establishes the classical form of the SFTWE and closely follows the argument in Luenberger (1994). Part 2 of the proof follows the extension of the argument in Debreu (1970) for the exchange case to the production case via the route taken in Smale (1974). Part 3 of the proof extends to the production the argument in Mas-Colell, Whinston and Green (1995; pp. 614 – 5) and Hara, Segal and Tadelis (1997; pg. 17-35).

Proof of Theorem 2

Since $\{w_n\}$ is an allocation sequence along the hyperplane $\{\omega : p\omega_n = p\omega_0\}$ such that $TL(\omega_n) \subset TL(\omega_{n-1})$ for $n \geq 1$, it follows that $E(\omega_n) \subseteq E(\omega_{n-1})$ for $n \geq 1$. This is so because the institution is a voluntary exchange, and equilibria, if they exist, must be in the upper-contour set of each consumer. The second property, that $E(\omega^*)$ has one element, $\{p,\omega^*\}$, the no-trade equilibrium allocation follows from Theorem 1.

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