INCOME EFFECTS, SUBSTITUTION EFFECTS, AND THE NUMBER FOUR

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1. Introduction

At some point in almost every course in Microeconomics there is a treatment of the effect on consumer demand of a change in commodity prices. Consumer reaction to a change in commodity prices is clearly a topic of central importance to economic theory and to the students' overall understanding of the operation of the price mechanism. Unfortunately, it is a piece of analysis rarely enjoyed by students because of the number of 'cases' which need to be considered, the apparent lack of unity in the subject matter and the seeming paucity of 'definitive conclusions' arrived at by the analysis.

The principal purpose of this paper is to show that there is a simple way to unify the analysis of a consumer's reaction to price changes in terms of 'fanning and curvature' of indifference curves. In the process we also hope to make more widely known and accessible an important result in modern consumer theory, namely, the Milleron-Mitiushin-Polterovich theorem (MMP). As we will see, the MMP result can also be interpreted in terms of the fanning and curvature of indifference curves.

2. Income effects, substitution effects and negatively sloped demand

As Dougan (1982) observes: "No proposition is more central to all of economics than the existence of an inverse relationship between the relative price of a commodity and the quantity of the commodity that people will purchase willingly." Dougan (1982; p. 809). Because of the centrality to economics of this proposition, almost all courses in Microeconomics include an analysis of the effect on consumer demand of a change in commodity prices. For familiar reasons, this analysis necessarily involves the introduction of the concepts of 'income effects' and 'substitution effects', usually with an attendant series of diagrams, lines of algebra and paragraphs of explanation to handle the cases where goods are ‘normal’, ‘inferior’ or ‘strongly inferior’. Once mastered, this analysis is very interesting, intuitive and instructive. The trouble is that many students never get to the point of appreciating the analysis, in part because the number of possible cases involved can obscure the essential unity of the analysis and also because the conclusions reached by the analysis are often heavily qualified. Also, the way income and substitution effect analysis is usually presented, there is no obvious connection between a consumers response to price changes and the underlying consumers preference ordering. We aim to show how that connection can be made with a view to unifying and simplifying the whole analysis.

To illustrate the point made above consider the following correct statement of the consequences for demand of a change in price: “… [if] the income effect is large enough, the total change in demand could be positive. This would mean that an increase in price could result in an *increase* in demand. This is the perverse Giffen case … [note] a Giffen good must be an inferior good. But an inferior good is not necessarily a Giffen good: the income effect not only has to be of the ‘wrong’ sign but it also has to be *large enough* to outweigh the ‘right’ sign of the substitution effect … [thus Giffen goods] not only have to be inferior goods, but they would have to be *very* inferior.” Varian (1987; pg. 140, emphasis added).

One question which this statement by Varian (and others like it in the literature)[1] often prompts from students is: ‘What exactly is a ‘large enough’ income effect and what does it mean for a good to be ‘very inferior’?’ In order to answer this question we first make three slightly technical observations. These observations include: (i) a definition of the ‘uncompensated law of demand’ (ULD); (ii) a recollection of the Slutsky equation; and (iii) a statement of the Milleron-Mitiushin-Polterovich theorem.

2.1 The uncompensated law of demand

Definition [ULD – Mas-Colell, Whinston and Green (1995; pg. 111)]: Consumer i’s demand function $x_i(p, M_i)$ obeys the *uncompensated law of demand* (ULD) if for any p, p? and M_i we have: $(p? - p)[x_i(p?, M_i) – (x_i(p, M_i)] = 0$, with strict inequality whenever $x_i(p?, M_i)$? $x_i(p, M_i)$ and p? ? p.

Remark: In the special case where just one commodity price changes, the ULD gives us a ‘downward sloping’ demand curve for a single commodity. It is of interest to discover circumstances in which the ULD holds.

2.2 The Slutsky equation

The Slutsky equation is a compact way of writing down algebraically what the geometry of income and substitution effects is trying to capture and is therefore a way of revealing what is needed for the ULD to hold. For a change in demand for commodity x with respect to its price p_x the Slutsky equation is as follows:

$$?x/?p_x = (?x/?p_x|_{U = constant}) – x \cdot ?x/?M \qquad (1)$$

Remark: The meaning of these terms is as follows: $?x/?p_x$ is the total effect on the (Marshallian) demand for good x of a change in its price; ($?x/?p_x|_{U = constant}$) is the substitution effect on demand for good x of a change in the price of good x; ?x/?M is the effect on demand for good x of a change in income and x is the amount of the good originally consumed. From the set up of equation (1) it is clear that $?x/?p_x$ will be negative provided that the income effect term x?x/?M does not 'dominate' the substitution effect term ($?x/?p_x|_{U = constant}$),. The issue for us is then to identify when the income and substitution effects combine to imply that $?x/?p_x < 0$. In particular we want to connect these circumstances with what's going on as far as the consumer's preferences are concerned, which is where the MMP theorem comes in.

2.3 The Milleron-Mitiushin-Polterovich theorem

The MMP theorem gives us the following criterion for preferences to imply the ULD.

Theorem (Milleron-Mitiushin-Polterovich): *Suppose that consumer i's preference ordering* $\preceq_i$ *is defined on the consumption possibility set* $X = \Re^L_+$. *Also suppose this preference ordering is representable by a concave utility function* $u_i(x)$ *that has continuous first and second order partial derivatives. If*

$$– \quad \{x_i[?^2u_i(x)/?x^2]x_i\}/x_i?u_i(x)/?x < 4 \qquad (2)$$

then consumer i's demand function, $x_i(p, M_i)$*, obeys the uncompensated law of demand.*

Remark: Commenting on this result, Mas-Colell, Whinston and Green (1995) remark that: "… for the ULD property to hold, the substitution effect (which is always well behaved) must be *large enough* to overcome possible 'perversities' coming from the wealth effects. The intriguing result [due to Milleron-Mitiushin-Polterovich] … gives a concrete expression to this relative dominance of the substitution effects." Mas-Colell, Whinston and Green (1995; pg. 112, emphasis added).

The authors invite the 'courageous reader' to attempt a proof of the theorem, an exercise which is, by their own admission, is non-trivial[2]. We do not propose to provide a proof of this result here[3]. Instead we want to explain and illustrate the condition presented in (2) in terms of 'fanning and curvature' of indifference curves[4]. By this means we intend to make the connection from income and substitution effects back to the consumer's underlying preference ordering.

3. The terms in expression (2)

At first sight expression (2) seems to contain some not particularly economically interpretable terms. On closer inspection however, we can see the following: (i) the term $?^2u_i(x)/?x^2 = [?^2u_{ij}/?x_j?x_k]$, is just a matrix of terms which express how consumer i's marginal utility of good j is influenced by a small change in the consumption of good k (including the case where k = j on the diagonal of the matrix). The value of the terms in this matrix is determined by the degree substitution that exists between goods. One way to picture that is interms of the degree of curvature in the consumer's indifference curves; (ii) the term $?u_i(x)/?x = (?u_{ij}/?x_j)$ is just a vector which lists the marginal utility of each good j = 1, n as far as consumer i is concerned. The marginal utility of a good is determined by how much utility changes as the consumption of the good changes i.e. as the consumer 'moves out' through their family of indifference curves. It is reasonably captured by the way in which the family of indifference curves generated by $u_i(x)$ 'fans out' in the space of possible consumptions. So what the MMP theorem is saying is that provided the ratio of these two quantities, weighted by the actual consumption bundle x_i, is of the right order for any initial consumption bundle (i.e. less than four), then the income effect will never dominate the substitution effect.

Some diagrams may make these remarks easier to understand. Consider the case depicted in Figure 1 where if the consumer's preferences are represented by the solid indifference curves. Then a price decrease [increase] for good 1 leads to a decrease [increase] in the consumption of good 1 (so the good is a classical Giffen good).

[**Figure 1 about here]**

It is clear from Figure 1 that the outcome occurs in part because of the way in which the consumer's indifference curves 'fan out' in the consumption space. If the indifference curves had been arranged differently (such as the broken indifference curve in Figure 1), then the demand curve for good one would have exhibited slope with respect to price.

The role of 'curvature' of the indifference curves in producing the Giffen (or conventional) demand behaviour can also be illustrated diagrammatically. Consider the situation in Figure 2a where the consumer doesn't regard two goods as being particularly good substitutes. As a consequence, for a given change in the relative price of the two goods there isn't a very big substitution effect. By contrast in Figure 2b the consumers indifference curves curve in such a way that for the same relative price change as in Figure 2a the substitution effect is much larger.

[Figure 2a about here] **[Figure 2b about here]**

Thus the degree of curvature of the consumer's indifference curves determines the strength of the substitution effect while the way in which the indifference curves fan out determines the sign and strength of the income effect. Consequently, whether a consumer's demand curve for a commodity slopes down (or up), depends on the relationship between the fanning and curvature of the indifference curves. A sufficient condition for this relationship to be 'right' and for the uncompensated law of demand to hold is provided by the Milleron-Mituishin-Polterovich theorem. A major purpose of this paper has been to explain that result in terms of the properties of a consumer's family of indifference curves and in the process to unify and simplify the standard income and substitution effect analysis.

3. Conclusion

At some point in almost every course in Microeconomics there is a treatment of the effect on a consumer's demand of a change in commodity prices. Unfortunately, this beautiful and important piece of analysis, generally summarized as 'income and substitution effect analysis', often leaves students a little bewildered, if not cold. The principal purpose of this paper has been to show that there is a simple way to unify the treatment of income and substitution effects in terms of 'fanning and curvature' of indifference curves. In the process of making this demonstration we also attempted to make accessible the Milleron-Mitiushin-Polterovich theorem as it is one of the more interesting advanced results in modern consumer theory, and a result that should be more widely known by students of economics. The paper will have achieved its objectives if income and substitution effect analysis ceases to be the burden it sometimes is for students and becomes instead an enlivening entree into a deeper investigation of consumer behaviour.

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FIGURES

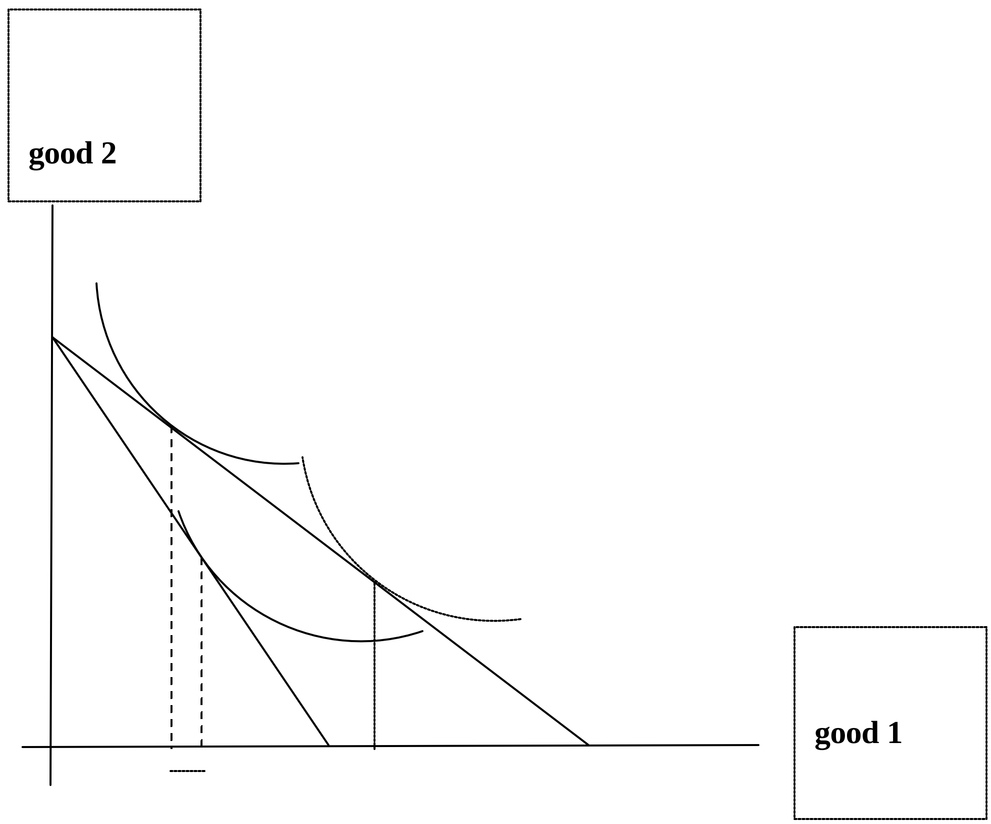

Figure 1

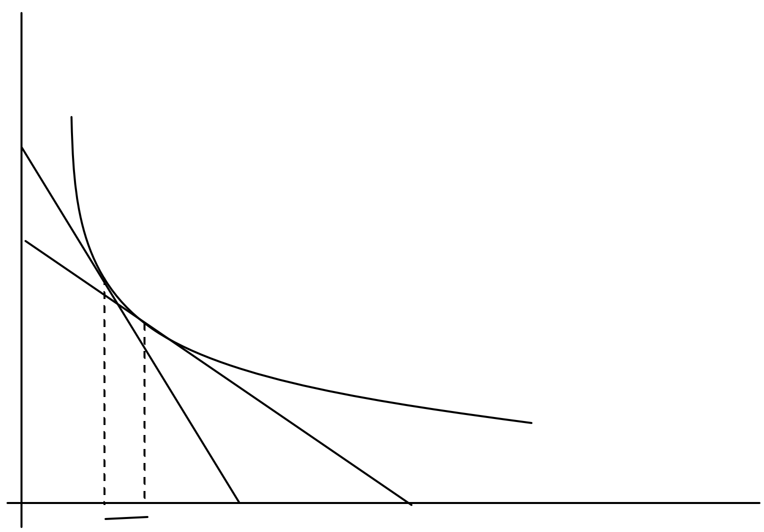

Figure 2a

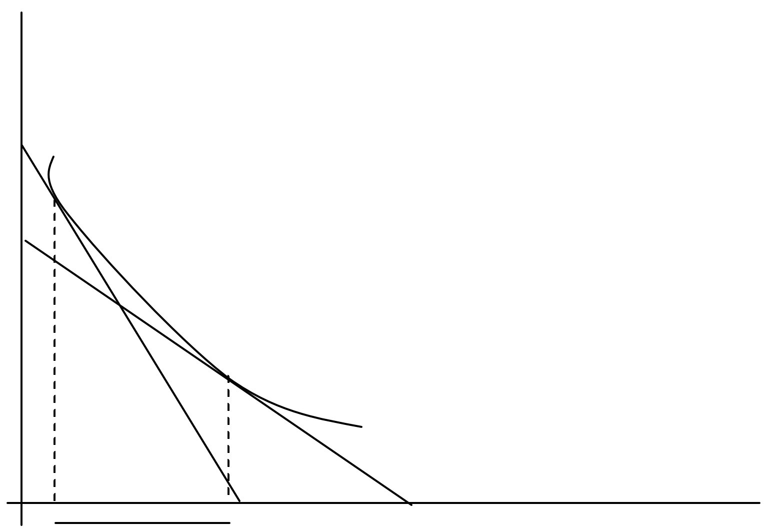

Figure 2b

ENDNOTES

[1] Similar remarks may be found in numerous places in the literature, for example: "Under stronger assumptions than those made here, Marshall deduced the so-called 'law of demand', that the slope of his demand curve with respect to price … is always negative. But it is now clear from the Slutsky equation that more than hypotheses 3.1-1 – 3.1-4 are required if this law is to remain valid … [indeed] there are three cases: (1) … commodity i is called a *superior good* … and this is enough to ensure that the Marshallian demand curve is downward sloping … (2) [c]ommodity i is an *inferior good* … even for inferior goods, the substitution effect may *still outweigh* the income effect [to ensure downward slopingness] … (3) … i is inferior and the income effect *overpowers* the substitution effect." Katzner (1970; pg. 59); "Concluding this section, it should be pointed out that only very weak properties of a consumer's market demand function have been established. In particular, without any further assumptions on preferences no general statement can be made about the direction of a change of the demand of a particular commodity when its price changes … If [the income effect] is positive (the case of a *superior good*) the own price effect is obviously negative. If the commodity is an *inferior good* … then the own-price effect *may be positive*." Barten and Bohm (1982; pg. 417, emphais added); "What can we say about the sign of (?x/?p)? The substitution effect T_2 is clearly negative … The income effect T_1, on the other hand is indeierminate in sign … Should it be negative, it would reinforce T_2; in that case an increase in P_x must decrease the purchase of x, and the demand curve of the utility-maximizing consumer would be negatively sloped. Should it be positive, but *relatibvely small in magnitude*, it would dilute the substitution effect, though the overall result would still be a downward-sloping demand curve. In the case T_1 is positive and *dominates* T_2 … then a rise in P_x will actually lead to a *larger* purchase of x …" Chiang (1984; pp. 407 – 8). "The first term on the right is the substitution effect, which, according to the above property, will always be nonpositive. The second term is the income effect. Usually this term will be negative (accounting for the minus sign), but for an inferior good it may change sign, and conceivably it could be *large enough* to dominate, leading to a Giffen good". Luenberger (1995; p. 151, emphasis added); "If the good is normal, the substitution and income effects of a price change are both negatively related to the price change and are therefore complementary. If the good is inferior, the income effect of a price change is positively related and the substitution effect is negatively related to the price change; the slope of the demand curve then depends on the *relative strengths* of the two effects". Eaton Eaton and Allen (2002; pg. 123).

[2] See Mas-Colell, Whinston and Green (1995; pg.113).

[3] A reasonably accessible proof is provided by Hara, Segal and Tadelis (1997; pg. 4-6).

[4] Recently Quah (2003) has made the following remark about the MMP criterion. Define the exprtession $?_u(x) = -\{x_i[?^2u_i(x)/?x^2]x_i\}/x_i?u_i(x)/?x$ and then: "[d]efining H(x + tx), where t is a scalar, one could check that $?_u(x)$ = -H''(0)/H'(0); in other words, $?_u(x)$ is a measure of the curvature of u at x and in the direction of x. The MMP theorem requires this value to be less than 4." Quah (2003; p. 714). We arrived at our insight that the MMP theorem imposes fanning and curvature requirements on the family of indifference curves independently of Quah (2003). In addition whereas Quah's characterization of the MMP result is in terms of the utility function, ours is in terms of the more transparent and accessible family of indifference curves.