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A Two Factor Model for Water Prices and Real Option Analysis

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Abstract

Standard real option models used in previous irrigation related studies may not be able to capture relevant impacts of temporary shocks (from water supply) and permanent shocks (from irrigated crop prices and changes in water policies) on water prices and may result in sup-optimal investment decisions. Permanent shocks lead to changes in long run water price which then gradually reverts towards its mean as irrigation land areas adjust. Temporary shocks lead to changes in short run water price which then reverts towards long run water price as the water buffer stock adjusts. In this paper, a two factor modelling framework is proposed to model this pattern of water prices. The model is used to analyse the optimal investment rule when such a composite water price process applies. It is found that high long run water prices have little impacts on the optimal investment decision while low long run water prices have important impacts. Consequently, using standard real option models that ignore long run water price may provide reasonable estimates of the optimal investment thresholds for high long run water price levels and usage of the computationally intensive two factor model is only necessary for low long run water prices. This result can be used to minimise computational effort in investment decision analysis.

Keywords: factor models, mean reversion, real option, water prices, investment.
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Introduction

Methods to determine optimal investment decisions are of significant interest in the irrigation sector. Environmental problems caused by diffused polluted return flows from irrigation farms are often addressed by cost-sharing policies that encourage efficient irrigation technology adoption, the design of which requires a good understanding of how private adoption decisions are made [Heaney et al., 2006]. To make an optimal choice between irrigation water purchase and water saving project investment to moderate the currently acute shortage of environmental water, the value and the decision rule for the investment project must first be determined.

It is now evident that the traditional net present value (NPV) criterion (invest when NPV is positive) cannot explain the investment decisions made in the private sector for investment projects with uncertain cash flows. Investment delay is often found when the NPV is positive and investment is observed only when the NPV is significantly high [Carey and Zilberman, 2002; Isik, 2006]. From the perspective of financial economic theory, investment delay can be optimal when investment cost is sunk and investment cash flow is stochastic. When these conditions hold, delaying investment allows the investor to observe future movements of the project value and benefit from upside risk (by investing) while avoid the downside risk (by not investing). The decision to invest is similar to the decision to exercise an American call option that pays dividend, and the American option exercise rule specified by financial economic theory can be used for the investment problem [Dixit and Pindyck, 1994]. Investment is optimal only when the expected NPV exceeds the investment
cost by the option value foregone when investing. The financial option model used to evaluate investment in real assets is often called the ‘standard real option model’. Application of the standard real option model has been vast [Dixit and Pindyck, 1994]. For irrigation related investment, the model has been applied to investigate the decision to reform a water property right system by Howitt [1995], the decision to adopt modern irrigation technologies by Carey and Zilberman [2002] and the decision to invest in irrigation dam by Michailidis and Mattas [2007].

Despite its ability to explain the observed investment behaviour, the standard real option model has restrictive assumptions that may affect quantitative results and policy recommendations. The most notable assumption is that the project value behaves according to either a geometric Brownian motion (GBM) or a standard Brownian motion, both of which are non-stationary. This assumption is essential for the closed form formula of the American call option value to be applicable. However, different from financial stock prices, commodity prices are subject to demand and supply forces and the project value should be stationary rather than non-stationary.

Several studies have departed from the standard option value model in an attempt to improve investment analysis. Baerenklau and Knapp [2007] examined the impacts of uncertainty on the irrigation technology adoption decision when both water prices and output prices are stationary. Isik [2006] compared the impacts of a GBM process and a mean reverting process on the investment appraisal. Using one stochastic process to model water price as in these studies, however, may not adequately describe the behaviour of water price.

Fundamental elements that determine the stochasticity of water price can be classified as temporary shocks and permanent shocks. Temporary shocks only affect water price for a short time. Examples of temporary shocks include weather variations in the
irrigation region that shift the irrigation water demand curve and variations in water inflows (to the storage reservoir in the region) that affect water supply. Long term irrigation capitals such as irrigation land areas take a long time to develop and are not likely to be altered by temporary shocks. Therefore, temporary shocks affect water demand and/or supply, but do not change the long term irrigation capital structure on either side.

Permanent shocks, on the other hand, are long lasting and induce changes in the irrigation capital structure. These shocks include output price changes that alter the expectation of far future output prices, changes in the management of water storage and changes in the policy that allocates water between the irrigation sector and other sectors. Different from temporary shocks, permanent shocks take time to generate their full impacts on water prices. For instance, irrigation output prices often have high degrees of autocorrelation and changes in output prices in one year are expected to affect output prices in many following years.

The irrigation capital structure, represented by the combination of irrigated annual crop land and perennial crop land, depends on the irrigation output prices and the distribution of water supply in the region. While the distribution of water supply can be treated as static, output prices are often more realistically modelled as mean reverting processes with low reversion rates [Deaton and Laroque, 1992]. Given an output price combination and an initial combination of irrigation land, the combination of irrigation land adjusts towards a member of a set rather than a single

\[1\text{ When water is stored across irrigation seasons, given a storage/release rule, the distribution of water release converges to a unique distribution (Moran, 1959). With a high convergence rate and a long time required to develop irrigation land, it is reasonable to treat equilibrium irrigation land depending only on the long run water release distribution.}\]
point, due to the sunk costs of irrigation land investments [Eberly and VanMieghem, 1997]. Changes in output prices move the set to new positions. As output prices revert to their long run levels, the set reverts to its corresponding long run position. A combination of existing land areas defines a long run water demand curve, which together with the expected rainfall and the expected irrigation water supply defines the long run water price level. With the combination of land areas reverting to its long run positions, long run water price reverts to its corresponding long run levels. Although it is more realistic to allow long run water price to have multiple reversion points, due to the multiple equilibria of irrigation land areas, this aspect will not be modelled in this paper. It is assumed that the long run water price follows a single ‘point’ reversion process².

Given a combination of irrigation land, under the impacts of temporary shocks of water supply and rainfalls and the management of water storage, short run water price follows a ‘point’ reverting process, with the point depending on the combination of land areas, or long run water price. As long run water price evolves, the point of reversion of short run water price evolves too. As a consequence, water price follows a point reverting process with the reverting point being stochastic and reverting to long run levels. This nature of stochasticity of water price has important implications for investment decisions and alters the choice of investment models used by analysts. The fact that there is a two factor price process also has implications. If the reversion rate of prices warrants a stochastic investment analysis to capture option value and if water price process can be better characterised as a two factor process than a single

² The concept of ‘point reverting’ is more general than ‘mean reverting’. In the Ornstein-Uhlenbeck process, the reversion point is also the mean of the variable. However, when the logarithm of water price follows the Ornstein-Uhlenbeck process, water price will revert to a point, but not to its mean.
factor process, then in principle a two factor process ought to be modelled and used in the analysis.

Multiple factor models have been examined in the energy economics literature. Pindyck [1999] estimated a long run, short run model in which the log of instantaneous price follows a mean reverting process and the log of long run price follows another mean reverting process. While the model is similar to the model formulated in this paper, Pindyck focused on estimation procedures, rather than the method to determine investment decisions. Schwartz and Smith [2000] examined the option to invest in a mining project where oil prices follow a composite process of a mean reverting process and a Brownian motion process. While the price of non-renewable resources can be reasonably modelled using random walk processes due to the unpredictability of new supply source discovery, prices of renewable resources such as water are often predictable to some extent and point reverting processes are more appropriate.

In this paper, a framework to determine optimal investment decisions when water price follows a two factor process is proposed. The framework is based on Schwartz and Smith [2000] model with the long run component being modelled by a mean reverting process, rather than a random walk process. With this framework, a closed form formula for expected water price can be derived, making the computation of investment project value easier and reducing computation time required for optimal investment computation. A closed form formula for European options on water price can also be derived, which can in turn be used to compute American options, using approximate valuation methods, e.g. see Geske and Johnson [1984]. Although these options are not yet introduced in water markets, there have been calls for their introduction to help irrigators manage water price risks [Cui and Schreider, 2009].
The paper has four sections. The theoretical model is outlined to model the water price process and closed functional forms for expected water prices and European call options written on water price are derived in the first section. In the second section, the method to assess the value of an investment option and optimal investment decision is described. Analysis of numerical results for a hypothetical set of parameters is provided in the third section.

1. Modelling framework

1.1 Water price process

Previous studies have modelled log of prices, instead of price levels, to follow diffusion processes so that prices are always positive [Pindyck, 1999; Schwartz and Smith, 2000]. When the log of short run price is modelled to revert to the log of long run price, as in Pindyck [1999], at the price level, short run price may not revert to long run price. This is because short run price is log-normally distributed and the expected value of a log-normal variate is an exponential of the sum of the corresponding normal variate mean plus half of the normal variate variance, rather than an exponential of the normal variate mean alone. To have short run water price reverting to long run water price and long run water price reverting to its mean, we model the log of short run price to long run price ratio and the log of long run price to long run mean ratio to follow mean reverting processes. When a suitable reversion point is chosen for the log of a price ratio, the price ratio will revert to one, which ensures that short run water price reverts to long run water price.

Let \( P(t) \) be the spot water price observed in water markets, \( P^*(t) \) be the long run water price that is observed through irrigated land areas, \( \alpha \) be the reversion point of long run water price, \( x(t) \) be the log of short run water price to long run water price
ratio, \( y(t) \) be the log of long run water price to the long run mean ratio. Then, short run water price will revert to long run water price and long run water price will revert to the long run mean if \( x(t) \) and \( y(t) \) follow the following mean reverting processes:

\[
dx(t) = \gamma(-\sigma_x^2/4\gamma - x(t))dt + \sigma_x dZ_x(t)
\]

(1)

\[
dy(t) = \kappa(-\sigma_y^2/4\kappa - y(t))dt + \sigma_y dZ_y(t)
\]

(2)

\[
x(t) = \ln\left(\frac{P(t)}{P^*(t)}\right)
\]

(3)

\[
y(t) = \ln\left(\frac{P^*(t)}{\alpha}\right).
\]

(4)

In the above equations, \( \gamma \) and \( \kappa \) are the reversion rates of \( x(t) \) and \( y(t) \), and \( \sigma_x^2 \) and \( \sigma_y^2 \) are the local variances of \( x(t) \) and \( y(t) \), respectively. As shown in section 1.3 below, \(-\sigma_x^2/4\gamma\) is the reversion point for \( x(t) \) that allows \( P(t) \) to revert to \( P^*(t) \) and \(-\sigma_y^2/4\kappa\) is the suitable reversion point for \( y(t) \). \( dZ_x(t) \) is a Weiner process that is equal to a white noise multiplied by the square root of time change, \( dZ_x(t) = \varepsilon_x \sqrt{dt} \), and \( dZ_y(t) \) is another Weiner process. Assume that the correlation between the short term change \( dZ_x(t) \) and long term change \( dZ_y(t) \) is constant:

\[
E(dZ_x dZ_y) = \rho dt.
\]

1.2 Expected water price and European option value

When water price follows processes described in Equations (1)-(4), closed functional forms for expected spot prices and European call option values can be derived. For the convenience of notation, lower case letters are used to denote the logarithm of price levels: \( p(t) = \ln P(t) \); \( p^*(t) = \ln P^*(t) \).
Expected spot price

Assume that interest rate is constant and agents are risk neutral\(^3\), once the expected water price can be determined, the computation of an invested project value is straightforward. Given the information observed at time \( t \), the expected water price at time \( u > t \) is given in Lemma 1.

**Lemma 1**

When water prices follow the composite process specified in Equations (1)-(4), the expected water price at time \( u \) conditional on the information observed at time \( t \), \((P(t), P^r(t))\) is:

\[
E_t [P(u)] = \exp\{\hat{\mu}(u) + \frac{1}{2} \hat{\sigma}^2(u)\},
\]

where:

\[
\hat{\mu}(u) = x(t)e^{-\gamma(u-t)} - \frac{\sigma_x^2}{4\gamma}(1-e^{-\gamma(u-t)}) + y(t)e^{-\kappa(u-t)} - \frac{\sigma_y^2}{4\kappa}(1-e^{-\kappa(u-t)}) + \ln \alpha
\]

and

\[
\hat{\sigma}^2(u) = \sigma_x^2 \frac{1-e^{-2\gamma(u-t)}}{2\gamma} + \sigma_y^2 \frac{1-e^{-2\kappa(u-t)}}{2\kappa} + 2\rho \sigma_x \sigma_y \frac{1-e^{-(\kappa+\gamma)(u-t)}}{\kappa+\gamma}.
\]

**Proof:** Appendix.

According to Lemma 1, when the future time \( u \) is equal to current time \( t \), the variance of the log of water price, \( \hat{\sigma}^2(u) \), reduces to zero and the mean of the log of water price \( \hat{\mu}(u) \) is equal to \( x(t) + y(t) + \ln \alpha \). As a result, the futures price is equal to the water price at time \( t \), \( P(t) \).

---

As the future time $u$ increases to infinity, the variance of $\ln P$, $\hat{\sigma}^2(u)$, increases and approaches a constant value:

$$\hat{\sigma}^2(u) \to \hat{\sigma}^2(\infty) = \sigma_x^2 / 2\gamma + \sigma_y^2 / 2\kappa + 2\rho\sigma_x\sigma_y / (\kappa + \gamma).$$

The mean of $\ln P$ also approaches a constant value, $\hat{\mu}(\infty) = -\frac{\sigma_x^2}{4\gamma} - \frac{\sigma_y^2}{4\kappa} + \ln \alpha$. The futures water price approaches a constant level which is a multiple of $\alpha$:

$$E_\tau(P) \to \alpha e^{\rho\sigma_x\sigma_y \kappa (\gamma + \kappa)}.$$

**European options on water price**

Consider a European call option written on water price with time to maturity $u - t$ and exercise price $K$. Since the option is only exercised if the water price at time $u$ exceeds $K$, the option value is the present value of the expected cash flow generated when the future spot water price exceeds the exercise price:

$$V_i(P(t), P^*(t), K) = e^{-r(u-t)}E_i[\max(P(u) - K, 0)],$$

where $r$ is the interest rate.

Based on the density function of the underlying asset (i.e. water price), the value of the European option can be calculated using the method given in *McDonald and Siegel* [1985]. The option value is given in Lemma 2.

**Lemma 2**

The current value (at date $t$) of a European call option with exercise price $K$ and maturity date $u$ is:

$$V_i(P(t), P^*(t), K) = e^{-r(u-t)}\{E_i[P(u)]^* N\left(\hat{\sigma} \frac{-\ln K - \hat{\mu}}{\hat{\sigma}}\right) - KN\left(\frac{\hat{\mu} - \ln K}{\hat{\sigma}}\right)\},$$

11
where:

\[ N[-] \text{ is the standard cumulative normal distribution} \]

\[ E_t[P(u)] \text{ is defined in Lemma 1} \]

\[
\hat{\mu}(u) = x(t)e^{-\gamma(u-t)} - \frac{\sigma_x^2}{4\gamma} (1 - e^{-\gamma(u-t)}) + y(t)e^{-\kappa(a-t)} - \frac{\sigma_y^2}{4\kappa} (1 - e^{-\kappa(u-t)}) + \ln \alpha
\]

\[
\hat{\sigma}^2(u) = \sigma_x^2 \frac{1 - e^{-2\gamma(u-t)}}{2\gamma} + \sigma_y^2 \frac{1 - e^{-2\kappa(u-t)}}{2\kappa} + 2\rho \sigma_x \sigma_y \frac{1 - e^{-(\kappa+\gamma)(u-t)}}{\kappa + \gamma}.
\]

**Proof:** Appendix.

It should be noted that although a closed functional form for an European option written on the value of the investment project is desirable, since it can be used in estimating the value of the real option, such closed functional form cannot be derived using the density function method when the log of water price follows mean reverting process. This is because the value of a project invested at a time \( t \) depends on the expected water prices in periods after \( t \) and when the log of water price follows mean reverting processes, the value of an investment project becomes a polynomial function of water prices at time \( t \). Consequently, the family to which the distribution of the project value belongs cannot be easily determined. This would not be a problem were water price levels modelled as mean reverting processes or were water prices modelled to follow geometric Brownian motions.

\[ ^4 \text{This becomes clear when the expected water prices given in Lemma 1 are used in Equation 10 below in calculating the value of an invested project.} \]
1.3 Reproducing one factor models

The two factor model specified in Equations (1)-(4) can be reduced to a single factor short run water price model or a single factor long run water price model by setting the relevant reversion rate to infinity.

Short run one factor model

Suppose the reversion rate $\kappa$ of the long run water price is set to infinity. Then, long run water price $P'(t)$ is always equal to its mean $\alpha$ and water price depends only on the parameters of the short run water price process. From Lemma 1, as $\kappa \to \infty$, the expected water price at a future time $u$ is:

$$E_r [P(u)] = \exp\{ \hat{\mu}(u) + \frac{1}{2} \hat{\sigma}^2(u) \}$$

where

$$\hat{\mu}(u) = x(t)e^{-\gamma(u-t)} - \frac{\sigma^2}{4\gamma} (1 - e^{-\gamma(u-t)}) + \ln \alpha$$

and

$$\hat{\sigma}^2(u) = \sigma^2_x \frac{1 - e^{-2\gamma(u-t)}}{2\gamma}.$$  \hspace{1cm} (5)

As the forecast horizon increases to infinity, $\hat{\mu}(u)$ approaches $-\frac{\sigma^2}{4\gamma} + \ln \alpha$ and $\hat{\sigma}^2(u)$ approaches $\sigma^2_x / 2\gamma$. The expected water price approaches $\alpha$ and short run water price reverts to the long run level.

Long run one factor model

In the two factor model, when the mean reversion rate of short run water price is infinitely large, from Lemma 1, the expected water price at time $u > t$ is:

$$E_r [P(u)] = \exp\{ \hat{\mu}(u) + \frac{1}{2} \hat{\sigma}^2(u) \},$$  \hspace{1cm} (7)
where \( \hat{\mu}(u) = y(t)e^{-\kappa(u-t)} - \frac{\sigma^2}{4\kappa}(1 - e^{-\kappa(u-t)}) + \ln \alpha \)  

and \( \sigma^2(u) = \sigma^2 \frac{1 - e^{-2\kappa(u-t)}}{2\kappa}. \)  

As the forecast horizon approaches infinity, the variance \( \hat{\sigma}^2(u) \) approaches \( \sigma^2 / 2\kappa \) and the mean \( \hat{\mu}(u) \) approaches \( -\frac{\sigma^2}{4\kappa} + \ln \alpha. \) The expected water price \( E_t[P(u)] \) approaches the long run level \( \alpha. \)

1.4 A hypothetical parameter set

A hypothetical set of parameters (Table 1) will be used to examine the properties of the optimal investment rule given by the two factor model. The parameter set is formed based on the following assumptions. The reversion rate of short run water price is assumed to be higher than the reversion rate of long run water price. The reversion rate of short run water price is expected to be high because water inflow to storage dam is highly variational while water storage capacity is often limited and once the water stock has been depleted or exceeded the dam capacity, short run water price achieves full reversion to long run water price. The reversion rate of long run water price is expected to be low due to the long time required to develop irrigation land and perennial trees and the long time required for irrigated crop prices to revert to their long run means.

Also, it is assumed that short run water price has a higher local variance compared to the local variance of long run water price. This is because irrigation water supply and rainfall in irrigation regions often have wide variations and transacted water prices in the short term often vary widely. In contrast, irrigated crop price variations are often
effectively reduced by buffer stocks and the resulted variation in long run water price is expected to be low.

Long run and short run water prices may be correlated since both are affected by changes in irrigated crop prices. The correlation between irrigated crop prices and long run water price, however, may be also low due to the long lag between a change in crop prices and a change in irrigation land area that causes a change in long run water price. It is therefore expected that the correlation between long run and short run water prices is low.

PLACE TABLE 1 HERE

2. Evaluation of real options

2.1 Example water investment problem

The optimal investment rule is examined for a simple investment project. The project is assumed to provide a constant water saving flow of one ML/year, for $T_k$ years. The investment decision is considered relevant (for initiation) for only $T$ years, after which, it is not considered anymore. In the case of irrigation technology, this assumption is reasonable when new, more efficient technologies that will become available in the future remain unknown at the current time. For public water savings project, the opportunity to construct (economically) a dam will pass if land that would be flooded by the reservoir is developed by other economic agents for residential or other uses or if land in the catchment is developed for uses that would make water storage permanently incompatible.

The investment problem can be expressed as:

$$\max_{0 \leq t \leq T} E \left\{ \int_t^{T_k} e^{-rn} P(u) - e^{-rT} K \mid (P(0), P^*(0)) \right\},$$

(10)
subject to (1)-(4).

The investment problem with both $T$ and $T_k$ infinitely large has been considered by Dixit and Pindyck (1994), with $T_k$ finite and $T$ infinitely large has been considered by Baerenklau and Knapp [2007], and with both $T$ and $T_k$ finite has been considered by Wu et al. [1994]. In the numerical example, it is assumed that $T = 10$ years; $T_k = 30$ years, the investment cost is $K = $800 and the interest rate is 5%.

### 2.2 Methods to evaluate American style options

Numerical methods for American style option evaluation can be used to evaluate investment option values. An American option can be evaluated using one of three approaches: finite difference method, simulation approach and lattice approach. Finite difference method relies on numerical approximations to find the value of state variables where the partial differential equation holds. For mean reverting processes, discretised state variables may be given negative probabilities, causing divergence in numerical results [Hull and White, 1990 p92]. Simulation approach is able to deal with a large number of state variables as well as different types of processes but is computationally intensive and does not lend itself easily to theoretical insights [Broadie and Glasserman, 1997; Cortazar et al., 2008; Longstaff and Schwartz, 2001]. The lattice approach is similar to the finite difference method, but deals directly with the probabilities of the up and the down movements of the discretised stochastic variables. This approach is more intuitive and can be used to solve complex real options where it is not even possible to write down the set of partial differential equations [Trigeorgis, 1991 p310]. In this paper, the binomial lattice approach described by Hahn and Dyer [2008] is used.
With the binomial lattice, each continuous time stochastic process is approximated with a discrete time binomial process. The time horizon is divided into equal steps and over each time step, the stochastic variable can either jump up or down by a fixed jump step. The jump step and the jump probabilities are chosen so that the first and second moments of the discrete time processes match those of the continuous time processes. The probabilities can be constrained to be between zero and one by a censoring scheme that sets probabilities to one if they exceed one and to zero if they are negative. As the time step approaches zero, the probabilities approach 0.5 and the discrete time processes approach the continuous time processes [Hahn and Dyer, 2008].

2.3 Computation of optimal investment rule

The investment problem is solved in discrete time. This problem is similar to job search problems considered by McCall [1970], Jovanovic [1979] and Neal [1999].

Let $\hat{T}_k$ and $\hat{T}$ be the number of time steps corresponding respectively to the useful life of the project $T_k$ and the life of the investment option $T$, when the length of each time step is $\Delta t$. The triplet $(t, x, y)$, where $t \in [0, \hat{T}]$, $x$ and $y$ are logarithms of price ratios, is an element of the state space. Given the state $(t, x, y)$, if the project is initiated, the NPV obtained is:

$$V(t, x, y) = \sum_{n=t}^{t+\hat{T}_k} e^{-nt} E\{P(u) \mid (t, x, y)\} \Delta t - e^{-nt} K,$$

where $E\{P(u) \mid (t, x, y)\}$ is calculated using Lemma 1.
Let $F(t, x, y)$ be the value of the option to invest given the state $(t, x_i, y_i)$. If the option is kept alive until the next period, the expected discounted value of the option will be:

$$e^{-r^N} E[F(t+1, x', y')] = e^{-r^N} \sum_{(x', y')} F(t+1, x', y') \Pr((x', y')|(x, y)),$$

where $\Pr((x', y')|(x, y))$ is the probability that state $(x', y')$ occurs in the next period conditional on the current state $(x, y)$. These transitional probabilities are calculated using the binomial lattice method discussed above. The expected discounted value of the option given by Equation (12) is called “deferral value” in the following.

At any node $(t, x_i, y_i)$, where $t < \hat{T}$, the option is exercised (the project is invested) if the value of immediate investment $V(t, x, y)$ exceeds the deferral value $e^{-r^N} E[F(t+1, x', y')]$. The value of the option at node $(t, x_i, y_i)$ is the maximum of the immediate investment value $V(t, x, y)$ and the deferral value $e^{-r^N} E[F(t+1, x', y')]$:

$$F(t, x, y) = \max\{V(t, x, y), e^{-r^N} E[F(t+1, x', y')]\}.$$  

(13)

After time $\hat{T}$, the option has no value:

$$F(\hat{T} + 1, x, y) = 0.$$  

(14)

The recursive Bellman equation (13) can be solved backwards, starting from period $\hat{T} + 1$. To reduce computational effort, Richardson extrapolation technique is used. The value of the real option is calculated for 20, 40, 60, 80 time steps per year and the obtained points are then fitted with a cubic polynomial. Using the estimated curve, the option value can be estimated for an infinitesimal time step. This extrapolation
technique has been shown to provide very accurate estimation of option values [Boyle et al., 1989].

3. Results

3.1 Impacts of trend on optimal investment

To provide insights into the behaviour of optimal investment in a two factor model, it is necessary to understand the impacts of trend versus uncertainty on investment decisions. To see the impacts of trend, consider a deterministic settings in which transacted water prices in future periods are assumed to follow the path of the expected short run water price given by the two factor model, conditional on the current short run and long run water prices. The mathematical expression of the optimization problem is simply Equation (10), which ignores the stochastic movements of water prices in future periods driven by stochastic processes (1)-(2). The interior solution $0 < \tau^* < T$ can be found from the first order conditions:

$$E_0\{P(\tau^*)\} = rK + e^{-r\tau^*} E_0\{P(\tau^* + T_K)\},$$

(15)

which states that at any considered investment time $\tau$, deferring investment by one period has a marginal cost equal to the value of one water unit at time $\tau$, and a marginal benefit equal to the sum of capital cost $rK$ and the value of a water unit provided with a delay of $T_K$ periods from time $\tau$. The value of the project obtained at $\tau^*$ must be compared with the values obtained from investment at $t = 0$ and investment at $t = T$ (corner solutions) to determine the globally optimal investment time. It should be noted that while the marginal cost of deferral, $E_0\{P(\tau^*)\}$, depends on the current water prices, $(P(0), P'(0))$, the marginal benefit can be considered as independent from current water prices because after the long time period $\tau^* + T_K$,
long run and short run water prices would have reverted to the long run mean and the expected water price $E_0\{P(\tau^* + T_k)\}$ is equal to the long run mean.

As illustrated in Figure 1, the marginal deferral cost curve is non-monotone and has its intercept determined by the current short run water price while its shape is determined by the current long run water price. When the current short run water price is above the marginal deferral benefit, low current long run water prices may result in multiple solutions to Equation (15) (Figure 1). In this case, either immediate investment or investing at the time $\tau^*$ at which the marginal cost is rising ($\tau_2^*$ in Figure 1) can be optimal, depending on the current long run water price. An increase in the current long run price will raise the value of immediate investment by a greater amount than the increase in the value of a future investment because the impacts of a long run price change on future prices decreases (due to mean reversion) as the time horizon increases. Therefore, a higher current long run price will favour immediate investment.

PLACE Figure 1 HERE

On the other hand, when the current short run water price is below the marginal deferral benefit, and the current long run water price is high, Equation (15) has only one solution because in the long term, the transacted water price reverts to a level above the marginal deferral benefit. Also, at time $t = 0$, the marginal deferral benefit is higher than the marginal deferral cost and the marginal deferral cost is rising, the value of the project increases as investment time increases. The interior solution to Equation (15) is also the optimal time to invest. The implication is that regardless of how high the current long run water price is, investment deferral is optimal. Immediate investment is only optimal when the current short run water price is above
the marginal benefit of deferral. Higher long run water prices will increase the value of investment while it is the short run water price that determines whether to defer or invest. The investment rule in this case is the same as the investment rule stipulated by a one factor model based on the short run water price.

PLACE Figure 2 HERE

3.2 Impacts of uncertainty on optimal investment

The investment thresholds given by the stochastic model and the deterministic model are depicted in Figure 3. The dollar amount next to each plotted point on the threshold of the stochastic model reports the value of the investment option given by that model while the percentage figures next to the threshold of the deterministic model reports the percentage loss of option value due to neglecting uncertainty impacts. For a given model, investment is optimal when the state is located on the right of the corresponding threshold curve while waiting is optimal when the state is on the left of the curve. Ignoring uncertainty results in a quite small loss of option value, indicating that a significant portion of the option value is due to trend. The investment threshold of the stochastic model is quite similar to the threshold given by the deterministic model. Increases in the current long run water price from a high level have little impacts on the investment threshold, just as in the deterministic case.

PLACE Figure 3 HERE

4. Conclusion

In this paper, a modelling framework has been outlined for water prices incorporating the impacts of temporary shocks and permanent shocks. Using a hypothetical set of parameters, the framework has been used to explore the impacts of long run and short run water prices on the value of irrigation technology investment option. It is found
that when long run water price is high, long run water price has little impacts on the optimal investment decision and the optimal investment rule is defined by a single threshold for short run water price, just as when a one factor model is used. Investing according to a one factor model may result in small losses as long as the threshold can be accurately estimated. However, when long run water price is low, both long run and short run water prices have important impacts on the optimal investment decision.

The single investment threshold given by one factor models cannot capture the two sources of information and will result in losses of investment value. These results suggest that to minimise computational efforts, analysts may only need to use the computationally intensive two factor model in identifying the investment thresholds for low levels of long run water price and can use one factor models for high levels of long run water price.
Appendix

Lemma 1

Given the water price follows the composite process specified in Equations (1) - (2) the expected water price at time $u$ conditional on the information observed at time $t$, $(P(t), P^*(t))$ is given by:

$$E[P(u)] = \exp\{\hat{\mu}(u) + \frac{1}{2} \hat{\sigma}^2(u)\},$$

where:

$$\hat{\mu}(u) = x(t)e^{-\gamma(u-t)} - \frac{\sigma^2_x}{2\gamma}(1-e^{-\gamma(u-t)}) + y(t)e^{-\kappa(u-t)} - \frac{\sigma^2_y}{2\kappa}(1-e^{-\kappa(u-t)}) + \ln \alpha$$

and

$$\hat{\sigma}^2(u) = \sigma^2_x \frac{1-e^{-2\gamma(u-t)}}{2\gamma} + \sigma^2_y \frac{1-e^{-2\kappa(u-t)}}{2\kappa} + 2\rho \sigma_x \sigma_y \frac{1-e^{-(\kappa+\gamma)(u-t)}}{\kappa+\gamma}.$$  

Proof:

From (3) and (4):

$$p(t) = x(t) + y(t) + \ln \alpha$$  

(A1)

Total differentiating (A1) gives:

$$dp(t) = dx(t) + dy(t).$$  

(A2)

Substituting (1) and (2) into (A2) gives:

$$dp(t) = -\sigma^2_x / 4 - \sigma^2_y / 4 - \kappa y(t) - \gamma x(t) \frac{dx(t)}{dt} + \sigma_y dZ_y(t) + \sigma_x dZ_x(t),$$

which says that $p(t)$ is normally distributed or $P(t)$ is log-normally distributed. The expected water price at time $u$ is determined if the mean and variance of $p(u)$ are known:

$$E[P(u)] = \exp\{E[p(u)] + \frac{1}{2} \text{Var}[p(u)]\}. \quad (A3)$$

$E[P(u)]$ then depends on the expected value of $x(u), p^*(u)$, their variance and covariance.

Applying the Ito’s Lemma$^5$ to functions $x(t)e^\gamma$ and $y(t)e^{\kappa}$, (1) and (2) can be expressed as:

$$x(u) = x(t)e^{-\gamma(u-t)} - \frac{\sigma^2_x}{4\gamma}(1-e^{-\gamma(u-t)}) + \sigma_x \int_t^u e^{-\gamma(u-s)} dZ_x(s) \quad (A4)$$

and

$$y(u) = y(t)e^{-\kappa(u-t)} - \frac{\sigma^2_y}{4\kappa}(1-e^{-\kappa(u-t)}) + \sigma_y \int_t^u e^{-\kappa(u-s)} dZ_y(s). \quad (A5)$$

$^5$ See Dixit and Pindyck (1994) for details on Ito Lemma for stochastic processes.
Taking expectation of (A4) and (A5) gives:

\[ E\{x(u)\} = x(t)e^{-\gamma(u-t)} - \frac{\sigma_x^2}{4\gamma} (1 - e^{-\gamma(u-t)}) \]  

(A6)

and \[ E\{y(u)\} = y(t)e^{-\kappa(u-t)} - \frac{\sigma_y^2}{4\kappa} (1 - e^{-\kappa(u-t)}) \].  

(A7)

From (A1), (A6) and (A7), \( E[p(u)] \) is:

\[ E[p(u)] = x(t)e^{-\gamma(u-t)} - \frac{\sigma_x^2}{4\gamma} (1 - e^{-\gamma(u-t)}) + y(t)e^{-\kappa(u-t)} - \frac{\sigma_y^2}{4\kappa} (1 - e^{-\kappa(u-t)}) + \ln \alpha \]  

(A8)

From (A2), (A4) and (A5), the variance of \( p(u) \) is:

\[
\text{var}[p(u)] = E[\{\sigma_x^u \int_t^u e^{-\gamma(s-t)} dZ_x(s) + \sigma_y^u \int_t^u e^{-\kappa(s-t)} dZ_y(s)\}^2] \\
= E[\sigma_x^u \int_t^u e^{-\gamma(s-t)} dZ_x(s)]^2 + 2E[\sigma_x^u \int_t^u e^{-\gamma(s-t)} dZ_x(s)]E[\sigma_y^u \int_t^u e^{-\kappa(s-t)} dZ_y(s)] + E[\sigma_y^u \int_t^u e^{-\kappa(s-t)} dZ_y(s)]^2
\]  

(A9)

The three components of (A9) are stochastic integrals. According to Bjerksund [1991] and Benhamou et al.[2008]:

\[ E[\int_t^u e^{-\gamma(s-t)} dZ_x(s)]^2 = \int_t^u e^{-2\gamma(s-t)} ds = e^{-2\gamma u} - \frac{e^{-2\gamma u}}{2\gamma} = \frac{1 - e^{-2\gamma(u-t)}}{2\gamma}, \]

\[ E[\int_t^u e^{-\kappa(s-t)} dZ_y(s)]^2 = \frac{1 - e^{-2\kappa(u-t)}}{2\kappa}, \]

and \[ E[\int_t^u e^{-\gamma(s-t)} dZ_x(s)]E[\int_t^u e^{-\kappa(s-t)} dZ_y(s)] = \int_t^u \rho e^{-(\kappa+\gamma)(u-s)} ds = \rho \frac{1 - e^{-(\kappa+\gamma)(u-t)}}{\kappa+\gamma}. \]

Then, (A9) becomes:

\[ \text{var}[p(u)] = \sigma_x^2 \frac{1 - e^{-2\gamma(u-t)}}{2\gamma} + \sigma_y^2 \frac{1 - e^{-2\kappa(u-t)}}{2\kappa} + 2\rho \sigma_x \sigma_y \frac{1 - e^{-(\kappa+\gamma)(u-t)}}{\kappa+\gamma}. \]  

(A10)

From (A3), (A8) and (A10), the expected water price at time \( u \) is:

\[ E[P(u)] = \exp\{\hat{\mu}(u) + \frac{1}{2} \hat{\sigma}^2(u)\}, \]  

(A11)

where:

\[ \hat{\mu}(u) = x(t)e^{-\gamma(u-t)} - \frac{\sigma_x^2}{4\gamma} (1 - e^{-\gamma(u-t)}) + y(t)e^{-\kappa(u-t)} - \frac{\sigma_y^2}{4\kappa} (1 - e^{-\kappa(u-t)}) + \ln \alpha \]
Lemma 2

The current value (at date $t$) of a European call option with exercise price $K$ and maturity date $u$ is:

$$V_t(P(t), P^*(t), K) = e^{-r(u-t)} [E_t[P(u)] - \ln K - \frac{\ln K - \hat{\mu}}{\hat{\sigma}} - K(\frac{\hat{\mu} - \ln K}{\hat{\sigma}})] ,$$

where:

$N[.]$ is the standard cumulative normal distribution

$E_t[P(u)]$ is defined in Lemma 1

$$\hat{\mu}(u) = x(t)e^{-y(u-t)}\tau_y - \frac{\sigma^2_x}{4\gamma}(1-e^{-2\gamma(u-t)}) + y(t)e^{-\kappa(u-t)} - \frac{\sigma^2_y}{4\kappa}(1-e^{-2\kappa(u-t)}) + \ln \alpha$$

$$\hat{\sigma}^2(u) = \sigma^2_x \frac{1-e^{-2\gamma(u-t)}}{2\gamma} + \sigma^2_y \frac{1-e^{-2\kappa(u-t)}}{2\kappa} + 2\rho\sigma_x\sigma_y \frac{1-e^{-(\kappa+\gamma)(u-t)}}{\kappa+\gamma}.$$

Proof:

According to the proof of Lemma 1, the log of water price at some future time $u$ is normally distributed with the mean and variance as:

$$\hat{\mu}(u) = E[p(u)]$$
and $$\hat{\sigma}^2(u) = \text{var}[p(u)].$$

where $E_t[p(u)]$ is given by (A8) and $\text{var}[p(u)]$ is given by (A10).

The density function of the $P(u)$ is:

$$g(P) = (P\hat{\sigma}\sqrt{2\pi})^{-1} \exp[-\frac{1}{2}(\frac{\ln P - \hat{\mu}}{\hat{\sigma}})^2]; \quad 0 < P < \infty.$$ 

Then:

$$E[\max(P(u) - K, 0)] = \int_k^\infty (P - K)g(P)dP. \quad (A12)$$

Let $z = \frac{\ln P - \hat{\mu}}{\hat{\sigma}}$ so that $P = e^{z\hat{\sigma} + \hat{\mu}}$, (A12) can be rewritten as:

$$E[\max(P(u) - K, 0)] = \int_{\frac{\ln K - \hat{\mu}}{\hat{\sigma}}}^\infty (e^{z\hat{\sigma} + \hat{\mu}} - K)(e^{z\hat{\sigma} + \hat{\mu}}\hat{\sigma}\sqrt{2\pi})^{-1} e^{\frac{1}{2}z^2} e^{z\hat{\sigma} + \hat{\mu}}\hat{\sigma}dz \quad (A13)$$

This can be rewritten as:

$$\int_{\frac{\ln K - \hat{\mu}}{\hat{\sigma}}}^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}z^2} e^{z\hat{\sigma} + \hat{\mu}}dz - K \int_{\frac{\ln K - \hat{\mu}}{\hat{\sigma}}}^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}z^2}dz \quad (A14)$$
Let \( w = z - \hat{\sigma} \), (A14) can be rewritten as:

\[
E[\max(P(u) - K, 0)] = e^{\hat{\mu} + \frac{1}{2} \hat{\sigma}^2} * \int_{\ln K / \hat{\sigma}}^{\infty} \int_{\ln K / \hat{\sigma}}^{\infty} e^{-\frac{1}{2} (z - \hat{\mu})^2} d\hat{\sigma}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} w^2} dw - K \int_{\ln K / \hat{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz \quad (A15)
\]

Let \( N(x) \) be the cumulative distribution function of the standard normal variable \( x \), then:

\[
N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz.
\quad (A16)
\]

Using (A16), (A15) can be written as:

\[
E[\max(P(u) - K, 0)] = e^{\hat{\mu} + \frac{1}{2} \hat{\sigma}^2} \left[1 - N\left(\frac{\ln K - \hat{\mu}}{\hat{\sigma}} - \hat{\sigma}\right)\right] - K[1 - N\left(\frac{\ln K - \hat{\mu}}{\hat{\sigma}}\right)] \quad (A17)
\]

Because the normal distribution is symmetric:

\[
1 - N(x) = N(-x),
\]

(A17) can be written as:

\[
E[\max(P(u) - K, 0)] = e^{\hat{\mu} + \frac{1}{2} \hat{\sigma}^2} \left[N\left(\hat{\sigma} - \frac{\ln K - \hat{\mu}}{\hat{\sigma}}\right)ight] - KN\left(\frac{\hat{\mu} - \ln K}{\hat{\sigma}}\right).
\]

According to (A3), \( e^{\frac{1}{2} \hat{\sigma}^2} = E[P(u)] \), the value of the European option is:

\[
E[\max(P(u) - K, 0)] = E[P(u)] * N\left(\hat{\sigma} - \frac{\ln K - \hat{\mu}}{\hat{\sigma}}\right) - KN\left(\frac{\hat{\mu} - \ln K}{\hat{\sigma}}\right)
\]
References


Table 1 Parameters for baseline analysis

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Figure 1 Marginal cost and benefit of investment deferral when current short run price is high and current long run price is low
Figure 2 Marginal cost and benefit of investment deferral when current short run price is low and current long run price is high
Figure 3 Investment threshold given by the two factor model